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Stability for time varying linear dynamic systems on time scales

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Abstract

We study conditions under which the solutions of a time varying linear dynamic system of the form $x^{\Delta}(t) = A(t)x(t)$ are stable on certain time scales. We give sufficient conditions for various types of stability, including Lyapunov-type stability criteria and eigenvalue conditions on "slowly varying" systems that ensure exponential stability. Finally, perturbations of the unforced system are investigated, and an instability criterion is also developed.

MSC: 93; 34; 39

Keywords: Stability; Time scale; Lyapunov; Time varying; Linear system; Slowly varying; Continuous; Discrete

1. Introduction

It is widely known that the stability characteristics of an autonomous linear system of differential or difference equations can be characterized completely by the placement of the eigenvalues of the system matrix [1,13]. Recently, Pötzsche et al. [23] authored a landmark paper which developed necessary and sufficient conditions for the stability of time invariant linear systems on arbitrary time scales. Their characterization included the sufficient condition that the eigenvalues of the system matrix be contained in the possibly disconnected set of stability $\mathscr{S}(\mathbb{T}) \subset \mathbb{C}^-$, which may change for each time scale on which the system is studied. The subsequent paper in [10] examined the stability characteristics of time varying and time invariant scalar dynamic equations on time scales and was the first paper to characterize the behavior of a time varying first order dynamic equation on arbitrary time scales.

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The intent of this paper is to extend the current results of autonomous linear dynamic systems to the more general case of nonautonomous linear dynamic systems on a large class of time scales (i.e. those time scales with bounded graininess and sup $\mathbb{T} = \infty$). We show that, in general, the placement of eigenvalues of the system matrix does not guarantee the stability or exponential stability of the time varying system, as is the case with autonomous linear systems of differential and difference equations [6,13,19,20,25] and dynamic equations on time scales [23]. We unify and extend the theorems of eigenvalue placement in the proper region of the complex plane for sufficiently slow varying system matrices of continuous and discrete nonautonomous systems, which yields exponential stability of the system, as in the classic papers [8,9,24], and the relatively recent paper [26]. To develop this theory for nonautonomous systems, we unify the theorems of uniform stability, uniform exponential stability, and uniform asymptotic stability for time varying systems by implementing a generalized time scales version of the "second (direct) method" of Lyapunov [22], a Russian mathematician and engineer, as in the standard papers on stability of continuous and discrete dynamical systems in [19,20].

In his dissertation of 1892, Lyapunov developed two methods for analyzing the stability of differential equations. His "second (direct) method" has become the most widely used tool for stability analysis of linear and nonlinear systems in both differential and difference equations. The idea is very straightforward and it involves measuring the energy of the system, usually the norm of the state variables, as the system evolves in time. The objective of the so-called "second (direct) method" of Lyapunov is this: *To answer questions of stability of differential and difference equations, utilizing the given form of the equations but without explicit knowledge of the solutions*. The principal idea of the second method is contained in the following physical reasoning: If the rate of change, $\mathrm{d}E(x)/\mathrm{d}t$, of the energy $E(x)$ of an isolated physical system is negative for every possible state x, except for a single equilibrium state x_{e}, then the energy will continually decrease until it finally assumes its minimum value $E(x_{\mathrm{e}})$. In other words, a system that is perturbed from its equilibrium state will always return to it. This is the intuitive concept of stability. It follows that the mathematical counterpart of the preceding statement is the following: *A dynamic system is stable* (*in the sense that it returns to equilibrium after any perturbation*) *if and only if there exists a* "*Lyapunov function,*" *i.e., some scalar function* $V(x)$ *of the state with the properties*: (*a*) $V(x) > 0$, $\dot{V}(x) < 0$, *when* $x \neq x_{\mathrm{e}}$, *and* (*b*) $V(x) = \dot{V}(x) = 0$ *when* $x = x_{\mathrm{e}}$ [19].

In engineering applications and applied mathematics problems, a solution usually is neither readily available nor easily calculated. As in adaptive control, which was born from a desire to stabilize certain classes of linear continuous systems without the need to explicitly identify the unknown system parameters, even a knowledge of the system matrix itself may not be fully available. The inherent beauty and elegance of the "second method" of Lyapunov is that knowledge of the exact solution is not necessary. The qualitative behavior of the solution to the system (i.e. the stability or instability) can be investigated without computing the actual solution.

By unifying and extending Lyapunov's "second method" to nonautonomous linear systems on time scales, we encounter the possibility of a time domain consisting of nonuniform distance between successive points. This proves to be a nontrivial issue and hence is seldom dealt with in the literature. It is, however, a rapidly rising theme in many engineering applications, such as the papers [16–18] which deal with high-gain adaptive controllers, digital systems, as well as very recent results from [11,12] which give new algorithms for adaptive controllers and bandwidth reduction using controller area networks. The time scale methods introduced and developed in this paper allow the examination and manipulation of the stability characteristics of dynamical systems without regard to the particular domain of the system, i.e. continuous, discrete, or mixed.

This paper is organized as follows. In Section 2, we give general definitions of our matrix norms, matrix definiteness, as well as stability definitions and characterizations so that the paper is reasonably self-contained. Section 3 introduces the unified theorems of uniform stability and uniform exponential stability of linear systems on time scales, as well as illustrations of these theorems in examples. Section 4 gives conditions on the eigenvalues of a sufficiently "slowly varying" system matrix which ensures exponential stability of the system solution. In Section 5, the stability properties of systems with perturbations are investigated. Finally, Section 6 demonstrates how the quadratic Lyapunov function developed in Section 3 can also be used to determine the instability of a system. We give a brief summary of the theory of time scales in the Appendix.

2. General definitions

We start by introducing definitions and notation that will be employed in the sequel.

The *Euclidean norm* of an $n \times 1$ vector $x(t)$ is defined to be a real-valued function of t and is denoted by

$$\|x(t)\| = \sqrt{x^{\mathrm{T}}(t)x(t)}.$$

The *induced norm* of an $m \times n$ matrix A is defined to be

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

The norm of A induced by the Euclidean norm above is equal to the nonnegative square root of the absolute value of the largest eigenvalue of the symmetric matrix $A^{\mathrm{T}}A$. Thus, we define this norm next. The *spectral norm* of an $m \times n$ matrix A is defined to be

$$\|A\| = \left[\max_{\|x\|=1} x^{\mathrm{T}}A^{\mathrm{T}}Ax\right]^{1/2}.$$

This will be the matrix norm that is used in the sequel and will be denoted by $\|\cdot\|$.

A symmetric matrix M is defined to be *positive semidefinite* if for all $n \times 1$ vectors x we have $x^{\mathrm{T}}Mx \geqslant 0$ and it is *positive definite* if $x^{\mathrm{T}}Mx \geqslant 0$, with equality only when $x = 0$. Negative semidefiniteness and definiteness are defined in terms of positive definiteness of $-M$.

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the regressive time varying linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}. \tag{2.1}$$

Definition 2.1. The time varying linear dynamic equation (2.1) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \leqslant \gamma \|x(t_0)\|, \quad t \geqslant t_0. \tag{2.2}$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (2.1), but also the asymptotic characteristics of the solutions as well. If the solutions to (2.1) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 2.2. The time varying linear dynamic equation (2.1) is called *uniformly exponentially stable* if there exist constants $\gamma,\ \lambda > 0$ with $-\lambda \in \mathscr{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \leqslant \|x(t_0)\|\gamma e_{-\lambda}(t, t_0), \quad t \geqslant t_0. \tag{2.3}$$

It is obvious by inspection of the previous definitions that we must have $\gamma \geqslant 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0.

The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 2.3. The linear state equation (2.1) is defined to be *uniformly asymptotically stable* if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\| \leqslant \delta\|x(t_0)\|, \quad t \geqslant t_0 + T. \tag{2.4}$$

It is noted that the time T that must pass before the norm of the solution satisfies (2.4) and the constant $\delta > 0$ is independent of the initial time t_0.

We now state and prove four theorems, the first three of which characterize uniform stability and uniform exponential stability in terms of the transition matrix for system (2.1). The fourth theorem illustrates the relationship between uniform asymptotic stability and uniform exponential stability.

Theorem 2.1. *The time varying linear dynamic equation* (2.1) *is uniformly stable if and only if there exists a* $\gamma > 0$ *such that*

$$\|\Phi_A(t, t_0)\| \leqslant \gamma$$

for all $t \geqslant t_0$ *with* $t, t_0 \in \mathbb{T}$.

Proof. Suppose that (2.1) is uniformly stable. Then, there is a $\gamma > 0$ such that for any $t_0,\ x(t_0)$, the solutions satisfy

$$\|x(t)\| \leqslant \gamma\|x(t_0)\|, \quad t \geqslant t_0.$$

Given any t_0 and $t_a \geqslant t_0$, let x_a be a vector such that

$$\|x_a\| = 1, \quad \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\|\,\|x_a\| = \|\Phi_A(t_a, t_0)\|$$

So the initial state $x(t_0) = x_a$ gives a solution of (2.1) that at time t_a satisfies

$$\|x(t_a)\| = \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\|\,\|x_a\| \leqslant \gamma\|x_a\|.$$

Since $\|x_a\| = 1$, we see that $\|\Phi_A(t_a, t_0)\| \leqslant \gamma$. Since x_a can be selected for any t_0 and $t_a \geqslant t_0$, we see that $\|\Phi_A(t, t_0)\| \leqslant \gamma$ for all $t, t_0 \in \mathbb{T}$.

Now suppose that there exists a γ such that $\|\Phi_A(t, t_0)\| \leqslant \gamma$ for all $t, t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (2.1) satisfies

$$\|x(t)\| = \|\Phi_A(t, t_0)x_0\| \leqslant \|\Phi_A(t, t_0)\| \, \|x_0\| \leqslant \gamma \|x_0\|, \quad t \geqslant t_0.$$

Thus, uniform stability of (2.1) is established. $\square$

Theorem 2.2. *The time varying linear dynamic equation* (2.1) *is uniformly exponentially stable if and only if there exist* $\lambda, \ \gamma > 0$ with $-\lambda \in \mathscr{R}^+$ *such that*

$$\|\Phi_A(t, t_0)\| \leqslant \gamma e_{-\lambda}(t, t_0)$$

for all $t \geqslant t_0$ *with* $t, t_0 \in \mathbb{T}$.

Proof. First suppose that (2.1) is exponentially stable. Then there exist $\gamma, \ \lambda > 0$ with $-\lambda \in \mathscr{R}^+$ such that for any t_0 and $x_0 = x(t_0)$, the solution of (2.1) satisfies

$$\|x(t)\| = \|x_0\| \gamma e_{-\lambda}(t, t_0), \quad t \geqslant t_0.$$

So for any t_0 and $t_a \geqslant t_0$, let x_a be a vector such that

$$\|x_a\| = 1, \quad \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \, \|x_a\| = \|\Phi_A(t_a, t_0)\|.$$

Then the initial state $x(t_0) = x_a$ gives a solution of (2.1) that at time t_a satisfies

$$\|x(t_a)\| = \|\Phi_A(t_a, t_0)x_a\| = \|\Phi_A(t_a, t_0)\| \|x_a\| \leqslant \|x_a\| \gamma e_{-\lambda}(t, t_0).$$

Since $\|x_a\| = 1$ and $-\lambda \in \mathscr{R}^+$, we have $\|\Phi_A(t, t_0)\| \leqslant \gamma e_{-\lambda}(t, t_0)$. Since x_a can be selected for any t_0 and $t_a \geqslant t_0$, we see that $\|\Phi_A(t, t_0)\| \leqslant \gamma e_{-\lambda}(t, t_0)$ for all $t, t_0 \in \mathbb{T}$.

Now suppose there exist $\gamma, \ \lambda > 0$ with $-\lambda \in \mathscr{R}^+$ such that $\|\Phi_A(t, t_0)\| \leqslant \gamma e_{-\lambda}(t, t_0)$ for all $t, t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (2.1) satisfies

$$\|x(t)\| \leqslant \|\Phi_A(t, t_0)x_0\| \leqslant \|\Phi_A(t, t_0)\| \|x_0\| \leqslant \|x_0\| \gamma e_{-\lambda}(t, t_0), \quad t \geqslant t_0,$$

and thus uniform exponential stability is attained. $\square$

Theorem 2.3. *Suppose there exists a constant* α *such that for all* $t \in \mathbb{T}$, $\|A(t)\| \leqslant \alpha$. *Then the linear state equation* (2.1) *is uniformly exponentially stable if and only if there exists a constant* β *such that*

$$\int_{\tau}^{t} \|\Phi_A(t, \sigma(s))\| \Delta s \leqslant \beta \tag{2.5}$$

for all $t, \tau \in \mathbb{T}$ *with* $t \geqslant \sigma(\tau)$.

Proof. Suppose that the state equation (2.1) is uniformly exponentially stable. By Theorem 2.2, there exist $\gamma, \ \lambda > 0$ with $-\lambda \in \mathscr{R}^+$ so that

$$\|\Phi_A(t, \tau)\| \leqslant \gamma e_{-\lambda}(t, \tau)$$

for all $t, \tau \in \mathbb{T}$ with $t \geqslant \tau$. So we now see that by a result in [5, p. 64, Theorem 2.39],

$$\begin{aligned}\int_\tau^t \|\Phi_A(t,\sigma(s))\|\Delta s &\leqslant \int_\tau^t \gamma e_{-\lambda}(t,\sigma(s))\Delta s\\ &= \frac{\gamma}{\lambda}[e_{-\lambda}(t,t) - e_{-\lambda}(t,\tau)]\\ &= \frac{\gamma}{\lambda}[1 - e_{-\lambda}(t,\tau)]\\ &\leqslant \frac{\gamma}{\lambda}\end{aligned}$$

for all $t \geqslant \sigma(\tau)$. Thus, we have established (2.5) with $\beta = \frac{\gamma}{\lambda}$.

Now suppose that (2.5) holds. We see that we can represent the state transition matrix as

$$\Phi_A(t,\tau) = I - \int_\tau^t [\Phi_A(t,s)]^{\Delta s}\Delta s = I + \int_\tau^t \Phi_A(t,\sigma(s))A(s)\Delta s,$$

so that, with $\|A(t)\| \leqslant \alpha$,

$$\|\Phi_A(t,\tau)\| \leqslant 1 + \int_\tau^t \|\Phi_A(t,\sigma(s))\|\,\|A(s)\|\Delta s \leqslant 1 + \alpha\beta$$

for all $t, \tau \in \mathbb{T}$ with $t \geqslant \sigma(\tau)$.

To complete the proof,

$$\begin{aligned}\|\Phi_A(t,\tau)\|(t-\tau) &= \int_\tau^t \|\Phi_A(t,\tau)\|\Delta s\\ &\leqslant \int_\tau^t \|\Phi_A(t,\sigma(s))\|\,\|\Phi_A(\sigma(s),\tau)\|\Delta s\\ &\leqslant \beta(1+\alpha\beta)\end{aligned} \tag{2.6}$$

for all $t \geqslant \sigma(\tau)$.

Now, choosing T with $T \geqslant 2\beta(1+\alpha\beta)$ and $t = \tau + T \in \mathbb{T}$, we obtain

$$\|\Phi_A(t,\tau)\| \leqslant \tfrac{1}{2}, \quad t,\tau \in \mathbb{T}. \tag{2.7}$$

Using the bound from Eqs. (2.6) and (2.7), we have the following set of inequalities on intervals in the time scale of the form $[\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$, with arbitrary τ:

$$\begin{aligned}\|\Phi_A(t,\tau)\| &\leqslant 1+\alpha\beta, \quad t \in [\tau, \tau+T)_{\mathbb{T}},\\ \|\Phi_A(t,\tau)\| &= \|\Phi_A(t,\tau+T)\Phi_A(\tau+T,\tau)\|\\ &\leqslant \|\Phi_A(t,\tau+T)\|\,\|\Phi_A(\tau+T,\tau)\|\\ &\leqslant \frac{1+\alpha\beta}{2}, \quad t \in [\tau+T, \tau+2T)_{\mathbb{T}},\\ \|\Phi_A(t,\tau)\| &= \|\Phi_A(t,\tau+2T)\Phi_A(\tau+2T,\tau+T)\Phi_A(\tau+T,\tau)\|\\ &\leqslant \|\Phi_A(t,\tau+2T)\|\|\Phi_A(\tau+2T,\tau+T)\|\|\Phi_A(\tau+T,\tau)\|\\ &\leqslant \frac{1+\alpha\beta}{2^2}, \quad t \in [\tau+2T, \tau+3T)_{\mathbb{T}}.\end{aligned}$$

In general, for any $\tau \in \mathbb{T}$, we have

$$\|\Phi_A(t,\tau)\| \leqslant \frac{1+\alpha\beta}{2^k}, \quad t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}.$$

We now choose the bounds to obtain a decaying exponential bound. Let $\gamma = 2(1+\alpha\beta)$ and define the positive (possibly piecewise defined) function $\lambda(t)$ (with $-\lambda(t) \in \mathscr{R}^+$) as the solution to $e_{-\lambda}(t,\tau) \geqslant e_{-\lambda}(\tau + (k+1)T, \tau) = \frac{1}{2^{k+1}}$, for $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ with $k \in \mathbb{N}_0$. Then for all $t, \tau \in \mathbb{T}$ with $t \geqslant \tau$, we obtain the decaying exponential bound

$$\|\Phi_A(t,\tau)\| \leqslant \gamma e_{-\lambda}(t,\tau).$$

Therefore, by Theorem 2.2, we have uniform exponential stability. □

For example, when $\mathbb{T} = \mathbb{R}$, the solution to

$$\mathrm{e}^{-\lambda(t-\tau)} \geqslant \mathrm{e}^{-\lambda(\tau+(k+1)T-\tau)} = \mathrm{e}^{-\lambda((k+1)T)} = \frac{1}{2^{k+1}}$$

with $k \in \mathbb{N}_0$ and $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ is $\lambda = -\frac{1}{T}\ln(\frac{1}{2})$.

When $\mathbb{T} = \mathbb{Z}$, the solution to

$$(1-\lambda)^{t-\tau} \geqslant (1-\lambda)^{\tau+(k+1)T-\tau} = (1-\lambda)^{(k+1)T} = \frac{1}{2^{k+1}}$$

with $k \in \mathbb{N}_0$ and $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ is $\lambda = 1 - (\frac{1}{2})^{-1/T}$, and $-\lambda \in \mathscr{R}^+$ on $\mathbb{T} = \mathbb{Z}$.

Theorem 2.4. *The linear state equation* (2.1) *is uniformly exponentially stable if and only if it is uniformly asymptotically stable*.

Proof. Suppose that system (2.1) is uniformly exponentially stable. This implies that there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathscr{R}^+$ so that $\|\Phi_A(t,\tau)\| \leqslant \gamma e_{-\lambda}(t,\tau)$ for $t \geqslant \tau$. Clearly, this implies uniform stability. Now, given a $\delta > 0$, we choose a sufficiently large positive constant $T \in \mathbb{T}$ so that $t_0 + T \in \mathbb{T}$ and $e_{-\lambda}(t_0 + T, t_0) \leqslant \frac{\delta}{\gamma}$. Then for any t_0 and x_0, and $t \geqslant T + t_0$ with $t \in \mathbb{T}$,

$$\begin{aligned}
\|x(t)\| &= \|\Phi_A(t,t_0)x_0\| \\
&\leqslant \|\Phi_A(t,t_0)\|\,\|x_0\| \\
&\leqslant \gamma e_{-\lambda}(t,t_0)\|x_0\| \\
&\leqslant \gamma e_{-\lambda}(t_0+T,t_0)\|x_0\| \\
&\leqslant \delta\|x_0\|, \quad t \geqslant t_0 + T.
\end{aligned}$$

Thus, (2.1) is uniformly asymptotically stable.

Now suppose the converse. By definition of uniform asymptotic stability, (2.1) is uniformly stable. Thus, there exists a constant $\gamma > 0$ so that

$$\|\Phi_A(t,\tau)\| \leqslant \gamma \quad \text{for all } t \geqslant \tau. \tag{2.8}$$

Choosing $\delta = \frac{1}{2}$, let T be a positive constant so that $t = t_0 + T \in \mathbb{T}$ and (2.4) is satisfied. Given a t_0 and letting x_a be so that $\|x_a\| = 1$, we have

$$\|\Phi_A(t_0+T,t_0)x_a\| = \|\Phi_A(t_0+T,t_0)\|.$$

When $x_0 = x_a$, the solution $x(t)$ of (2.1) satisfies

$$\|x(t)\| = \|x(t_0+T)\| = \|\Phi_A(t_0+T,t_0)x_a\| = \|\Phi_A(t_0+T,t_0)\|\,\|x_a\| \leqslant \tfrac{1}{2}\|x_a\|.$$

From this, we obtain

$$\|\Phi_A(t_0+T,t_0)\| \leqslant \tfrac{1}{2}. \tag{2.9}$$

It is easy to see that for any t_0 there exists an x_a as claimed. Therefore, the above inequality holds for any t_0. Thus, by using (2.8) and (2.9) exactly as in Theorem 2.3, uniform exponential stability is obtained. □

3. Stability of the time varying linear dynamic system

In this section, we investigate the stability of the regressive time varying linear dynamic system of the form

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}. \tag{3.1}$$

Our goal is to assess the stability of the unforced system by observing the system's total energy as the state of the system evolves in time. If the total energy of the system decreases as the state evolves, then the state vector approaches a constant value (equilibrium point) corresponding to zero energy as time increases. The stability of the system involves the growth characteristics of solutions of the state equation, and these properties can be measured by a suitable (energy-like) scalar function of the state vector. In the following two subsections, we discuss the boundedness properties and asymptotic behavior as $t \to \infty$ of solutions of system (3.1). The present issue is obtaining a proper scalar function.

We assume that the time scale $\mathbb{T}$ is unbounded above. To start, we consider conditions that imply all solutions of the linear state equation (3.1) are such that $\|x(t)\|^2 \to 0$ as $t \to \infty$. For any solution of (3.1), the delta derivative of the scalar function

$$\|x(t)\|^2 = x^{\mathrm{T}}(t)x(t)$$

with respect to t is:

$$\begin{aligned}
&[\|x(t)\|^2]^{\Delta_t} \\
&\quad = x^{\mathrm{T}^{\Delta}}(t)x(t) + x^{\mathrm{T}^{\sigma}}(t)x^{\Delta}(t) \\
&\quad = x^{\mathrm{T}}(t)A^{\mathrm{T}}(t)x(t) + x^{\mathrm{T}}(t)(I+\mu(t)A^{\mathrm{T}}(t))A(t)x(t) \\
&\quad = x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t)]x(t).
\end{aligned} \tag{3.2}$$

So if the quadratic form we obtained is negative definite, i.e. $A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t)$ is negative definite at each t, then $\|x(t)\|^2$ will decrease monotonically as t increases. We later show that if there exists a $v > 0$ so that $A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t) \leqslant -vI$ for all t, then $\|x(t)\|^2 \to 0$ as $t \to \infty$. To formalize our discussion, we define time-dependent quadratic forms that are useful for analyzing stability. We will refer to these quadratic forms as *unified time scale quadratic Lyapunov functions*. For a symmetric matrix $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ we write the general quadratic Lyapunov function as $x^{\mathrm{T}}(t)Q(t)x(t)$. If $x(t)$ is a solution to (3.1), and since $x^{\mathrm{T}}(t)Q(t)x(t)$ has a scalar output, our interest lies in the behavior of the quantity $x^{\mathrm{T}}(t)Q(t)x(t)$ for $t \geqslant t_0$. With this we now define one of the main ideas of this paper.

Definition 3.1. Let $Q(t)$ be a symmetric matrix such that $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A *unified time scale quadratic Lyapunov function* is given by

$$x^{\mathrm{T}}(t)Q(t)x(t), \quad t \geqslant t_0, \tag{3.3}$$

with delta derivative

$$\begin{aligned}[x^{\mathrm{T}}(t)Q(t)x(t)]^{\Delta_t} &= x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t)Q(t)\\ &\quad + (I+\mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t)+Q(t)A(t)+\mu(t)Q^{\Delta}(t)A(t))]x(t)\\ &= x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)+\mu(t)A^{\mathrm{T}}(t)Q(t)A(t)\\ &\quad + (I+\mu(t)A^{\mathrm{T}}(t))Q^{\Delta}(t)(I+\mu(t)A(t))]x(t).\end{aligned}$$

The matrix dynamic equation that is obtained by differentiating (3.3) with respect to t is given by

$$\begin{aligned}&A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)+\mu(t)A^{\mathrm{T}}(t)Q(t)A(t)\\ &\quad + (I+\mu(t)A^{\mathrm{T}}(t))Q^{\Delta}(t)(I+\mu(t)A(t)) = -M, \qquad M=M^{\mathrm{T}}.\end{aligned}$$

One can easily see that it merges with the familiar continuous matrix differential equation ($\mathbb{T}=\mathbb{R}$) and discrete ($\mathbb{T}=\mathbb{Z}$) difference (recursive) equation obtained from the respective quadratic Lyapunov functions in $\mathbb{R}$ and $\mathbb{Z}$.

For the continuous case, we observe that $\mu(t) \equiv 0$ when $\mathbb{T}=\mathbb{R}$. Thus, from (3.1) we now have the continuous system

$$\dot{x}(t) = A(t)x(t), \quad t \geqslant t_0. \tag{3.4}$$

The derivative of the quadratic Lyapunov function that emerges from (3.4) is

$$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)Q(t)x(t)] = x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)+\dot{Q}(t)]x(t),$$

where

$$A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)+\dot{Q}(t) = -M, \quad M=M^{\mathrm{T}},$$

is the familiar matrix differential equation [6,8,19,24,25] derived from the continuous system (3.4).

For the discrete case $\mathbb{T}=\mathbb{Z}$, we note that systems of difference equations in $\mathbb{Z}$ are traditionally written in recursive form

$$x(t+1) = A_R(t)x(t), \quad t \geqslant t_0, \tag{3.5}$$

while the difference form is written

$$x^{\Delta}(t) = \Delta x(t) = x(t+1)-x(t) = A(t)x(t), \quad t \geqslant t_0. \tag{3.6}$$

Thus, changing from difference form to recursion just requires a unit shift on the matrix $A(t)$, that is,

$$x(t+1) = (I+A(t))x(t) = A_R(t)x(t),$$

where $A_R = (I+A)$.

Now we see that with the unified time scale quadratic Lyapunov function above, noting that when in $\mathbb{Z}$, $\mu(t) \equiv 1$, we obtain

$$\begin{aligned}
&x^{\mathrm{T}}(t)[A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + A^{\mathrm{T}}(t)Q(t)A(t) + (I + A^{\mathrm{T}}(t))\Delta Q(t)(I + A(t))]x(t)\\
&\quad = x^{\mathrm{T}}(t)[(A_R^{\mathrm{T}}(t) - I)Q(t) + Q(t)(A_R(t) - I) + (A_R^{\mathrm{T}}(t) - I)Q(t)(A_R(t) - I)\\
&\qquad + A_R^{\mathrm{T}}(t)\Delta Q(t)A_R(t)]x(t)\\
&\quad = x^{\mathrm{T}}(t)[A_R^{\mathrm{T}}(t)Q(t) - Q(t) + Q(t)A_R(t) - Q(t) + A_R^{\mathrm{T}}(t)Q(t)A_R(t)\\
&\qquad - A_R^{\mathrm{T}}(t)Q(t) - Q(t)A_R(t) + Q(t) + A_R^{\mathrm{T}}(t)\Delta Q(t)A_R(t)]x(t)\\
&\quad = x^{\mathrm{T}}(t)[-Q(t) + A_R^{\mathrm{T}}(t)Q(t)A_R(t) + A_R^{\mathrm{T}}(t)\Delta Q(t)A_R(t)]x(t)\\
&\quad = x^{\mathrm{T}}(t)[-Q(t) + A_R^{\mathrm{T}}(t)Q(t)A_R(t) + A_R^{\mathrm{T}}(t)(Q(t+1) - Q(t))A_R(t)]x(t)\\
&\quad = x^{\mathrm{T}}(t)[A_R^{\mathrm{T}}(t)Q(t+1)A_R(t) - Q(t)]x(t),
\end{aligned}$$

where

$$A_R^{\mathrm{T}}(t)Q(t+1)A_R(t) - Q(t) = -M, \quad M = M^{\mathrm{T}},$$

is the well-known discrete matrix recursion equation [9,20,25] for the recursive system (3.5).

This shows that the unified time scale matrix dynamic equation merges into the continuous and discrete cases easily because of the time varying graininess $\mu(t)$. This unified time scale matrix dynamic equation not only unifies the two special cases of continuous and discrete time, it also extends these notions for arbitrary time scales $\mathbb{T}$, and as such plays a crucial role in our analysis.

3.1. *Uniform stability*

In this section, we introduce the criteria for uniform stability of system (3.1). The criteria introduced in Theorem 3.1 is a generalization of the Lyapunov criteria for uniform stability of discrete and continuous linear systems that can be found in the famous papers in [19,20]. Uniform stability involves the boundedness of all solutions of system (3.1) and in the following theorem we derive sufficient conditions for uniform stability of the system. The strategy is to state requirements on the matrix $Q(t)$ so that the corresponding quadratic form yields uniform stability of the system.

Theorem 3.1. *The time varying linear dynamic system* (3.1) *is uniformly stable if for all* $t \in \mathbb{T}$, *there exists a symmetric matrix* $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that*

(i) $\eta I \leqslant Q(t) \leqslant \rho I$,
(ii) $A^{\mathrm{T}}(t)Q(t) + (I + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leqslant 0$,
where $\eta, \rho \in \mathbb{R}^+$.

Proof. For any t_0 and $x(t_0) = x_0$, by (ii) and [5, Definition 1.71],

$$x^{\mathrm{T}}(t)Q(t)x(t) - x^{\mathrm{T}}(t_0)Q(t_0)x(t_0) = \int_{t_0}^{t} [x^{\mathrm{T}}(s)Q(s)x(s)]^{\Delta_s}\Delta s \leqslant 0$$

for $t \geqslant t_0$. Using (i),

$$\eta\|x(t)\|^2 \leqslant x^{\mathrm{T}}(t)Q(t)x(t) \leqslant x^{\mathrm{T}}(t_0)Q(t_0)x(t_0) \leqslant \rho\|x(t_0)\|^2,$$

which implies

$$\|x(t)\| \leqslant \sqrt{\frac{\rho}{\eta}}\|x(t_0)\|.$$

Since this last statement holds for all t_0 and $x(t_0) = x_0$, Eq. (3.1) is uniformly stable. □

To illustrate this theorem, we present an example.

Example 3.1. Consider the time varying linear dynamic system

$$x^{\Delta}(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix} x(t),$$

where $a(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ for all $t \in \mathbb{T}$. Choose $Q(t) = I$, so that $x^{\mathrm{T}}(t)Q(t)x(t) = x^{\mathrm{T}}(t)x(t) = \|x(t)\|^2$. In Theorem 3.1, (i) is satisfied when $\eta = \rho = 1$. To satisfy the second requirement, we see for $Q(t) = I$, $Q^{\Delta}(t) = 0$ so

$$A^{\mathrm{T}}(t)Q(t) + (I + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leqslant 0$$

becomes

$$A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t) \leqslant 0.$$

Now

$$A(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix}, \quad A^{\mathrm{T}}(t) = \begin{bmatrix} -2 & -1 \\ 1 & -a(t) \end{bmatrix}$$

and

$$\mu(t)A^{\mathrm{T}}(t)A(t) = \mu(t) \begin{bmatrix} 5 & a(t) - 2 \\ a(t) - 2 & a(t)^2 + 1 \end{bmatrix},$$

so

$$A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t) = \begin{bmatrix} 5\mu(t) - 4 & (a(t) - 2)\mu(t) \\ (a(t) - 2)\mu(t) & (a(t)^2 + 1)\mu(t) - 2a(t) \end{bmatrix}.$$

For any 2×2 matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

to be negative semidefinite, we need $-m_{11}, -m_{22} \geqslant 0$ and $\det(M) \geqslant 0$. For our matrix

$$A^*(t) := A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t),$$

we need

$$\begin{aligned} &-a^*_{11} = 4 - 5\mu(t) \geqslant 0 \text{ which implies } 0 \leqslant \mu(t) \leqslant \tfrac{4}{5}, \\ &-a^*_{22} = -((a(t)^2 + 1)\mu(t) - 2a(t)) \geqslant 0 \end{aligned}$$

and

$$\det(A^*(t)) = 4\mu(t)^2a(t)^2 - 4\mu(t)a(t)^2 + 4\mu(t)^2a(t) - 10\mu(t)a(t) + 8a(t) + \mu(t)^2 - 4\mu(t) \geqslant 0.$$

It is easy to confirm that for each $0 \leqslant \mu(t) \leqslant \frac{4}{5}$, the interval in which $-a^*_{22} \geqslant 0$ always contains the interval in which $\det(A^*(t)) \geqslant 0$. Thus, we only need to concern ourselves with the latter inequality. If $\mu(t) = \frac{4}{5}$, the only possible value that the function $a(t)$ may be is 2. If we let $\mu(t) = \frac{1}{2}$, we see that a window emerges for the allowable values of the function $a(t) : \frac{1}{2} \leqslant a(t) \leqslant \frac{7}{2}$. Letting $\mu(t) = \frac{2}{5}$, we see that another window develops for the allowable values of the function $a(t) : \frac{1}{3} \leqslant a(t) \leqslant \frac{9}{2}$. It is quite interesting to note that as $\mu(t) \to 0$, the window opens up to infinite length, bounded below by 0. Therefore, when $\mathbb{T} = \mathbb{R}$, the only requirement for $a(t)$ is that it is positive for all $t \in \mathbb{T}$.

3.2. Uniform exponential stability

We now introduce sufficient criteria for uniform exponential stability of system (3.1). The criteria introduced in Theorem 3.2 is again a generalization of the Lyapunov criteria for uniform exponential stability of discrete and continuous linear systems, which can be found in the companion papers in [19,20], as well as the classic text by Hahn [13]. There is a slight, but very powerful variation from uniform stability to uniform exponential stability. By requiring $Q(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ to be symmetric, positive definite, and bounded above and below by positive definite matrices, along with a strictly negative definite delta derivative, i.e.

$$[x^{\mathrm{T}}(t)Q(t)x(t)]^{\Delta} \leqslant -\varepsilon x^{\mathrm{T}}(t)x(t)$$

for some $\varepsilon > 0$, we will show that all solutions of (3.1) are bounded above by a decaying exponential and go to zero as $t \to \infty$. Uniform exponential stability does imply that system (3.1) is uniformly stable, but the converse is not true.

Theorem 3.2. *The time varying linear dynamic system* (3.1) *is uniformly exponentially stable if there exists a symmetric matrix* $Q(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that for all* $t \in \mathbb{T}$

(i) $\eta I \leqslant Q(t) \leqslant \rho I$
(ii) $A^{\mathrm{T}}(t)Q(t) + (I + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leqslant -\nu I$,
where $\eta, \rho, \nu \in \mathbb{R}^+$ *and* $\frac{-\nu}{\rho} \in \mathscr{R}^+$.

Proof. For any initial condition t_0 and $x(t_0) = x_0$ with corresponding solution $x(t)$ of (3.1), we see that for all $t \geqslant t_0$, (ii) yields

$$[x^{\mathrm{T}}(t)Q(t)x(t)]^{\Delta} \leqslant -\nu\|x(t)\|^2.$$

Also, for all $t \geqslant t_0$, (i) implies

$$x^{\mathrm{T}}(t)Q(t)x(t) \leqslant \rho\|x(t)\|^2.$$

Thus

$$[x^{\mathrm{T}}(t)Q(t)x(t)]^{\Delta} \leqslant \frac{-\nu}{\rho} x^{\mathrm{T}}(t)Q(t)x(t),$$

for all $t \geqslant t_0$. Since $\frac{-\nu}{\rho} \in \mathscr{R}^+$, we can employ the time scale version of Gronwall's inequality [5] to obtain

$$x^{\mathrm{T}}(t)Q(t)x(t) \leqslant x^{\mathrm{T}}(t_0)Q(t_0)x(t_0)e_{\frac{-\nu}{\rho}}(t, t_0), \quad t \geqslant t_0. \tag{3.7}$$

By (i), $\eta I \leqslant Q(t)$ which is equivalent to $\eta\|x(t)\|^2 \leqslant x^{\mathrm{T}}(t)Q(t)x(t)$ and division by η along with (3.7) yields

$$\|x(t)\|^2 \leqslant \frac{1}{\eta} x^{\mathrm{T}}(t)Q(t)x(t) \leqslant \frac{1}{\eta} x^{\mathrm{T}}(t_0)Q(t_0)x(t_0)e_{\frac{-\nu}{\rho}}(t, t_0), \quad t \geqslant t_0.$$

Since $x^{\mathrm{T}}(t_0)Q(t_0)x(t_0) \leqslant \rho\|x(t_0)\|^2$, this implies

$$\|x(t)\|^2 \leqslant \frac{\rho}{\eta}\|x(t_0)\|^2 e_{\frac{-\nu}{\rho}}(t, t_0),$$

which yields

$$\|x(t)\| \leqslant \|x(t_0)\| \sqrt{\frac{\rho}{\eta} e_{\frac{-\nu}{\rho}}(t, t_0)}, \quad t \geqslant t_0.$$

This holds for arbitrary t_0 and $x(t_0)$. Thus, uniform exponential stability is obtained. □

We present another example to show the difference between uniform and exponential stability.

Example 3.2. Consider again the time varying linear dynamic system

$$x^{\Delta}(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix} x(t),$$

where we now let $a(t) = \sin(t) + 2$ which is obviously in $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ for all $t \in \mathbb{T}$. We note that $\sin(t)$ is the usual sine function that gives the sine value of each point in $\mathbb{T}$ and it is *not* the time scale function $\sin_1(t, 0)$.

Again, choose $Q(t) = I$, so that $x^{\mathrm{T}}(t)Q(t)x(t) = x^{\mathrm{T}}(t)x(t) = \|x(t)\|^2$. In Theorem 3.1, (i) is satisfied when $\eta = \rho = 1$. To satisfy the second requirement, we see $Q(t) = I$, so $Q^{\Delta}(t) = 0$ and thus

$$A^{\mathrm{T}}(t)Q(t) + (I + \mu(t)A^{\mathrm{T}}(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leqslant -\nu I$$

becomes

$$A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t) \leqslant -\nu I.$$

For any 2×2 matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

to be negative definite, we need $-m_{11} > 0$ and $\det(M) > 0$. For our matrix

$$A^*(t) := A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t),$$

we need $-a^*_{11} = 4 - 5\mu(t) > 0$ which implies that $0 \leqslant \mu(t) < \frac{4}{5}$ and

$$\begin{aligned}\det(A^*(t)) &= 4\sin^2(t)\mu(t)^2 + 20\,\sin(t)\mu(t)^2 + 25\mu(t)^2\\ &\quad - 4\sin^2(t)\mu(t) - 26\,\sin(t)\mu(t) - 40\mu(t) + 8\,\sin(t) + 16 > 0.\end{aligned}$$

We note that $\det(A^*(t)) > 0$ for all $t \in \mathbb{T}$ as long as $0 \leqslant \mu(t) < \frac{1}{2}$.

For instance, letting $\mathbb{T} = \mathbb{P}_{.6,.4} = \bigcup_{k=0}^{\infty}[k, k+.6]$, in this time scale

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty}[k, k+.6) \\ .4 & \text{if } t \in \bigcup_{k=0}^{\infty}\{k+.6\} \end{cases}$$

Here, $\mu(t) \leqslant \frac{1}{2}$ for all $t \in \mathbb{T}$. From the previous example, we see that the allowable values are $\frac{1}{2} < a(t) < \frac{7}{2}$, which is satisfied for all $t \in \mathbb{T}$.

For any t, the eigenvalues of the matrix $A^*(t)$ have a maximum value less than $-\frac{1}{2}$ when $\mu(t) < \frac{1}{2}$. As $\mu(t)$ decreases to 0, the maximum value decreases. Therefore, the maximum of all of the eigenvalues of the matrix $A^*(t)$ is less than $-\frac{1}{2}$. So $A^*(t)$ is negative definite. Thus, we can set $v = \frac{1}{2}$.

Checking that $-\frac{v}{\rho} = -\frac{1}{2} \in \mathscr{R}^+$, we now know that the norm of any solution $x(t)$ with initial value $x(t_0)$ is bounded above by the always positive decaying time scale exponential function $\|x(t_0)\|\sqrt{e_{-\frac{1}{2}}(t, t_0)}$. By letting $Q(t) = I$, the matrix $A^*(t)$ meets the criteria (i), (ii) in Theorem (3.2). Thus, the system above is uniformly exponentially stable.

3.3. *Finding the matrix $Q(t)$*

First, we give a closed form for the unique, symmetric, and positive definite solution matrix to the *time scale Lyapunov matrix equation*

$$A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + \mu(t)A^{\mathrm{T}}(t)Q(t)A(t) = -M. \tag{3.8}$$

Remark. We note that the time scale Lyapunov matrix equation is the unification (with $B(t) \equiv A^{\mathrm{T}}(t)$) of the Sylvester matrix equation [3]

$$XA(t) + B(t)X = -M$$

for the case $\mathbb{T} = \mathbb{R}$ and the Stein equation

$$B(t)XA(t) - X = -M$$

for the case $\mathbb{T} = \mathbb{Z}$. The Stein matrix equation above is written assuming that one is using recursive form. It can easily be transformed into an equivalent difference form

$$XA(t) + B(t)X + B(t)XA(t) = -M.$$

To prove that the matrix $Q(t)$ is a solution to the time scale Lyapunov matrix equation (3.8), we first state the following theorem and corollary that can be found in [5].

Theorem 3.3. *Suppose* $A \in \mathscr{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ *and* C *is differentiable. If* C *is a solution of the matrix dynamic equation*

$$C^{\Delta} = A(t)C - C^{\sigma}A(t)$$

then

$$C(t)e_A(t,s) = e_A(t,s)C(s).$$

Corollary 3.1. *Suppose* $A \in \mathscr{R}$ *and* C *is a constant matrix. If* C *commutes with* $A(t)$*, then* C *commutes with* $e_{A(t)}$*. In particular, if* $A(t)$ *is constant matrix with respect to* $e_{A(t)}$*, then* $A(t)$ *commutes with* $e_{A(t)}$*.*

Now we present one of the main results of the paper.

Theorem 3.4. *If the* $n \times n$ *matrix* $A(t)$ *has all eigenvalues in the corresponding Hilger circle for every* $t \geqslant t_0$*, then for each* $t \in \mathbb{T}$*, there exists some time scale* $\mathbb{S}$ *such that integration over* $I :=[0,\infty)_{\mathbb{S}}$ *yields a unique solution to* (3.8) *given by*

$$Q(t) = \int_I e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)\Delta s. \tag{3.9}$$

Moreover, if M *is positive definite, then* $Q(t)$ *is positive definite for all* $t \geqslant t_0$*.*

Proof. First, we fix an arbitrary $t \in \mathbb{T}$. Since all eigenvalues of $A(t)$ are in the corresponding Hilger circle, [23] shows (3.9) converges, so that $Q(t)$ is well defined. We now show for each fixed $t \in \mathbb{T}$, $Q(t)$ is a solution of (3.8).

Case I: $\mu(t) > 0$. Since $\mu(t)$ is a positive number, we define the time scale $\mathbb{S} = \mu(t)\mathbb{N}_0$. So for each $s \in \mathbb{S}$, we have that $\mu(s) \equiv \mu(t)$; in other words, $\mathbb{S}$ has constant graininess. Now substituting (3.9) in the following with integration over $I = [0,\infty)_{\mathbb{S}}$ we obtain

$$\begin{aligned}
&A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + \mu(t)A^{\mathrm{T}}(t)Q(t)A(t)\\
&\quad= \int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)\Delta s\\
&\qquad + \int_I e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s\\
&\qquad + \mu(t)\int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s\\
&\quad= \int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)[I + \mu(t)A(t)]\Delta s\\
&\qquad + \int_I e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s\\
&\quad= \int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)M[I + \mu(t)A(t)]e_{A(t)}(s,0)\Delta s\\
&\qquad + \int_I e_{A^{\mathrm{T}}(t)}(s,0)MA(t)e_{A(t)}(s,0)\Delta s.
\end{aligned}$$

Since $\mu(t)=\mu(s)$, we continue with the last line as

$$
\begin{aligned}
&= \int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)M[I+\mu(s)A(t)]e_{A(t)}(s,0)\Delta s\\
&\quad + \int_I e_{A^{\mathrm{T}}(t)}(s,0)MA(t)e_{A(t)}(s,0)\Delta s\\
&= \int_I [e_{A^{\mathrm{T}}(t)}(s,0)]^{\Delta_s} Me^{\sigma}_{A(t)}(s,0)\Delta s\\
&\quad + \int_I e_{A^{\mathrm{T}}(t)}(s,0)M[e_{A(t)}(s,0)]^{\Delta_s}\Delta s\\
&= \int_I [e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)]^{\Delta_s}\Delta s\\
&= [e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)]|_0^{\infty}\\
&= -M.
\end{aligned}
$$

Case II: $\mu(t)=0$. Since $\mu(t)=0$, we define the time scale $\mathbb{S}=\mathbb{R}$. Now substituting (3.9) in the following with integration over $I=[0,\infty)$ we obtain,

$$
\begin{aligned}
&A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)+\mu(t)A^{\mathrm{T}}(t)Q(t)A(t)\\
&\quad = A^{\mathrm{T}}(t)Q(t)+Q(t)A(t)\\
&\quad = \int_I A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)\Delta s + \int_I e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s\\
&\quad = \int_I A^{\mathrm{T}}(t)e^{A^{\mathrm{T}}(t)\cdot s}Me^{A(t)\cdot s}\,\mathrm{d}s + \int_I e^{A^{\mathrm{T}}(t)\cdot s}Me^{A(t)\cdot s}A(t)\,\mathrm{d}s\\
&\quad = \int_I \frac{\mathrm{d}}{\mathrm{d}s}[e^{A^{\mathrm{T}}(t)\cdot s}]Me^{A(t)\cdot s} + e^{A^{\mathrm{T}}(t)\cdot s}M\frac{\mathrm{d}}{\mathrm{d}s}[e^{A(t)\cdot s}]\,\mathrm{d}s\\
&\quad = \int_I \frac{\mathrm{d}}{\mathrm{d}s}[e^{A^{\mathrm{T}}(t)\cdot s}Me^{A(t)\cdot s}]\,\mathrm{d}s\\
&\quad = [e^{A^{\mathrm{T}}(t)\cdot s}Me^{A(t)\cdot s}]|_0^{\infty}\\
&\quad = -M.
\end{aligned}
$$

Since $t\in\mathbb{T}$ was arbitrary, but fixed, we see that $Q(t)$ defined as in (3.9) is a solution of (3.8) for each $t\in\mathbb{T}$. Now, to show that $Q(t)$ is unique, suppose that $Q^*(t)$ is another solution to (3.8). Then

$$A^{\mathrm{T}}(t)[Q^*(t)-Q(t)]+[Q^*(t)-Q(t)]A(t)+\mu(t)A^{\mathrm{T}}(t)[Q^*(t)-Q(t)]A(t)=0,$$

which implies

$$
\begin{aligned}
&e_{A^{\mathrm{T}}(t)}(s,0)A^{\mathrm{T}}(t)[Q^*(t)-Q(t)]e_{A(t)}(s,0)+e_{A^{\mathrm{T}}(t)}(s,0)[Q^*(t)-Q(t)]A(t)e_{A(t)}(s,0)\\
&\quad +\mu(t)e_{A^{\mathrm{T}}(t)}(s,0)A^{\mathrm{T}}(t)[Q^*(t)-Q(t)]A(t)e_{A(t)}(s,0)=0, \quad s\geqslant 0.
\end{aligned}
$$

From this we obtain

$$[e_{A^{\mathrm{T}}(t)}(s,0)[Q^*(t)-Q(t)]e_{A(t)}(s,0)]^{\Delta_s}=0, \quad s\geqslant 0. \tag{3.10}$$

Integrating both sides of (3.10) over $[0, \infty)_{\mathbb{S}}$, we have

$$[e_{A^{\mathrm{T}}(t)}(s,0)[Q^*(t)-Q(t)]e_{A(t)}(s,0)]|_0^{\infty} = -(Q^*(t)-Q(t)) = 0,$$

which implies that $Q^*(t)=Q(t)$.

Lastly, suppose that M is positive definite. Recall that M positive definite implies $x^{\mathrm{T}}Mx>0$, for all $n \times 1$ vectors $x \neq 0$. Clearly, $Q(t)$ is symmetric. To prove that $Q(t)$ is positive definite, we notice that for any nonzero $n \times 1$ vector $x(t)$,

$$x^{\mathrm{T}}(t)Q(t)x(t) = \int_I x^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)Me_{A(t)}(s,0)x(t)\Delta s > 0$$

which is true since M is positive definite. Hence, $Q(t)$ is positive definite. □

4. Slowly varying systems

The placement of eigenvalues in the complex plane of a time invariant matrix is a necessary and sufficient condition to ensure the stability and/or exponential stability of the system. This is a well-known fact in the theory of differential equations and difference equations, and it is investigated in depth in the landmark paper on the stability of time invariant linear systems on time scales in [23].

However, eigenvalue placement alone is neither necessary nor sufficient in the general case of any time varying linear dynamic system. Texts such as [6,7,25] give examples of time varying systems with "frozen" (time invariant) eigenvalues with negative real parts as well as bounded system matrices that still exhibit instability. The classic papers [8,24], and a recent paper [26] demonstrate this fact for systems of differential equations, but they do show that under certain conditions, such as a bounded and sufficiently slowly varying system matrix, exponential stability can be obtained with correct eigenvalue placement in the complex plane. Desoer also published a similar paper [9] (a discrete analog to [8]) which illustrates the same instability characteristic of time varying systems in the discrete setting, but remedies the situation in essentially the same manner, with a bounded and sufficiently slow varying system matrix.

To begin, we state a definition from [23], in which the stability region for time invariant linear systems on time scales is introduced. This definition essentially says if the time average of the constant $\lambda \in \mathbb{C}$ is negative and $1+\mu(t)\lambda \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, then λ resides in the *regressive set of exponential stability* $\mathscr{S}(\mathbb{T})$, defined below. This definition is an integral part of the requirement for exponential stability of a time invariant linear system on an arbitrary time scale. If $\lambda_i \in \mathscr{S}(\mathbb{T})$ for all $i=1,\ldots,n$, and are uniformly regressive (see Appendix), then system (2.1), with $A(t)\equiv A$ constant, is uniformly exponentially stable, (i.e. there exists an $\alpha>0$ such that for any $t_0 \in \mathbb{T}$, $\gamma>0$ can be chosen independently of t_0 such that $\|\Phi_A(t,t_0)\| \leqslant \|x(t_0)\|\gamma e^{-\alpha(t-t_0)}$).

Definition 4.1 (*Pötzsche et al.* [23]). The *regressive set of exponential stability* for the dynamic system (2.1) when $A(t)\equiv A$ is a constant is defined to be the set

$$\mathscr{S}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{\mathrm{T}} \lim_{\searrow \mu(\tau)} \frac{\log|1+s\lambda|}{s} \Delta\tau < 0 \right\}.$$

The regressive set of exponential stability is contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda)<0\}$ at all times. The reader is referred to [23] for more explanation.

In the main theorem that follows, we require the eigenvalues $\lambda_i(t)$ of the time varying matrix $A(t)$ to reside in the corresponding Hilger circle for all $t \geqslant t_0$ and $i = 1, \dots, n$. We note that the Hilger circle is defined as the set

$$\left\{\lambda \in \mathbb{C} : \left|\frac{1}{\mu(t)} + \lambda(t)\right| < \frac{1}{\mu(t)}\right\} \subset \mathscr{S}(\mathbb{T}).$$

Finally, we introduce the definition of the Kronecker product for use in Theorem 4.1. The Kronecker product allows the multiplication of any two matrices, regardless of the dimensions. This operation is an integral part of the theorem since it offers an unusual way to represent a matrix equation as a vector valued equation from which we can easily obtain bounds on the solution matrix. Some useful properties are given in Lemma 4.1.

Definition 4.2. The *Kronecker product* of the $n_A \times m_A$ matrix A and the $n_B \times m_B$ matrix B is the $n_A n_B \times m_A m_B$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m_A}B \\ \vdots & \ddots & \vdots \\ a_{n_A 1}B & \cdots & a_{n_A m_A}B \end{bmatrix}. \tag{4.1}$$

Lemma 4.1 (*Zhang [27]*). *Assume $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ with complex valued entries.*

(i) $(A \otimes I_n)(I_m \otimes B) = A \otimes B = (I_m \otimes B)(A \otimes I_n)$.
(ii) *If λ_i and γ_j are the eigenvalues for A and B, respectively, with $i = 1, \dots, m$ and $j = 1, \dots, n$, then the eigenvalues of $A \otimes B$ are*

$$\lambda_i \gamma_j, \quad i = 1, \dots, m, \ \ j = 1, \dots, n,$$

and the eigenvalues of $(A \otimes I_n) + (I_m \otimes B)$ are

$$\lambda_i + \gamma_j, \quad i = 1, \dots, m, \ \ j = 1, \dots, n.$$

We now present the theorem for uniform exponential stability of slowly time varying systems which involves an eigenvalue condition on the time varying matrix $A(t)$ as well as the requirement that $A(t)$ is norm bounded and varies at a sufficiently slow rate (i.e. $\|A^{\Delta}(t)\| \leqslant \beta$, for some positive constant β and all $t \in \mathbb{T}$).

Theorem 4.1 (*Exponential stability for slowly time varying systems*). *Suppose for the regressive time varying linear dynamic system* (3.1) *with $A(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ we have $\mu_{\max}, \ \mu^{\Delta}_{\max} < \infty$, there exists a constant $\alpha > 0$ such that $\|A(t)\| \leqslant \alpha$, and there exists a constant $0 < \varepsilon < \frac{1}{\mu_{\max}} \leqslant \frac{1}{\mu(t)}$ such that for every pointwise eigenvalue $\lambda_i(t)$ of $A(t)$, $\mathrm{Re}_{\mu}[\lambda_i(t)] \leqslant -\varepsilon < 0$. Then there exists a $\beta > 0$ such that if $\|A^{\Delta}(t)\| \leqslant \beta$,* (3.1) *is uniformly exponentially stable.*

Proof. For each $t \in \mathbb{T}$, let $Q(t)$ be the solution of

$$A^{\mathrm{T}}(t)Q(t) + Q(t)A(t) + \mu(t)A^{\mathrm{T}}(t)Q(t)A(t) = -I. \tag{4.2}$$

By Theorem 3.4, existence, uniqueness, and positive definiteness of $Q(t)$ for each t is guaranteed. We also note that for each $t \in \mathbb{T}$, the solution of (4.2) is

$$Q(t) = \int_I e_{A^{\mathrm{T}}(t)}(s,0) e_{A(t)}(s,0)\Delta s,$$

where $I := [0,\infty)_{\mathbb{S}}$ and $\mathbb{S} = \mu(t)\mathbb{N}_0$. For the remaining part of the proof, we show that $Q(t)$ can be used to satisfy the requirements of Theorem 3.2, so that uniform exponential stability of (3.1) follows. First, we use the Kronecker product and some of its properties to show the boundedness of the matrix $Q(t)$. We let v_i denote the ith column of I, and $q_i(t)$ denote the ith column of $Q(t)$. We then define the $n^2 \times 1$ vectors

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad q(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix}.$$

It can be computed to confirm that the $n \times n$ matrix equation (4.2) can be written as the $n^2 \times 1$ vector equation

$$[(A^{\mathrm{T}}(t) \otimes I) + (I \otimes A^{\mathrm{T}}(t)) + \mu(t)(A^{\mathrm{T}}(t) \otimes A^{\mathrm{T}}(t))]q(t) = -v. \tag{4.3}$$

We now prove that $q(t)$ is bounded above and that there exists a positive constant ρ such that $Q(t) \leqslant \rho I$, for all $t \in \mathbb{T}$. Since $A(t) \in \mathscr{R}$, this implies that the pointwise eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ of $A(t)$ are also regressive. We also note that $I \in \mathscr{R}$. The pointwise eigenvalues of $A^{\mathrm{T}}(t) \otimes I$ and $I \otimes A^{\mathrm{T}}(t)$ are also $\lambda_1(t), \ldots, \lambda_n(t)$, by previously mentioned properties of the Kronecker product in Lemma 4.1. Because $(\mathscr{R}(\mathbb{T}, \mathbb{R}^{n^2 \times n^2}), \oplus)$ is a group we have that $(A^{\mathrm{T}}(t) \otimes I),\ (I \otimes A^{\mathrm{T}}(t)) \in \mathscr{R}$ yields

$$\begin{aligned}
&(A^{\mathrm{T}}(t) \otimes I) \oplus (I \otimes A^{\mathrm{T}}(t)) \\
&\quad = (A^{\mathrm{T}}(t) \otimes I) + (I \otimes A^{\mathrm{T}}(t)) + \mu(t)(A^{\mathrm{T}}(t) \otimes I)(I \otimes A^{\mathrm{T}}(t)) \\
&\quad = (A^{\mathrm{T}}(t) \otimes I) + (I \otimes A^{\mathrm{T}}(t)) + \mu(t)(A^{\mathrm{T}}(t) \otimes A^{\mathrm{T}}(t)) \in \mathscr{R}
\end{aligned}$$

for all $t \in \mathbb{T}$.

Now, we show that $(A^{\mathrm{T}}(t) \otimes I) \oplus (I \otimes A^{\mathrm{T}}(t))$ has no eigenvalues equal to zero, so that $\det[(A^{\mathrm{T}}(t) \otimes I) \oplus (I \otimes A^{\mathrm{T}}(t))] \neq 0$. The n^2 pointwise eigenvalues of $(A^{\mathrm{T}}(t) \otimes I) \oplus (I \otimes A^{\mathrm{T}}(t)) = (A^{\mathrm{T}}(t) \otimes I) + (I \otimes A^{\mathrm{T}}(t)) + \mu(t)(A^{\mathrm{T}}(t) \otimes A^{\mathrm{T}}(t))$ are:

$$\lambda_{i,j}(t) = \lambda_i(t) \oplus \lambda_j(t) = \lambda_i(t) + \lambda_j(t) + \mu(t)\lambda_i(t)\lambda_j(t) \in \mathscr{R}$$

for all $i, j = 1, \ldots, n$.

Recall that since $\mathrm{Re}_\mu[\lambda_i(t)] \leqslant -\varepsilon$ we have that $|1+\mu(t)\lambda_i(t)|<1$. Observe

$$\begin{aligned}\mathrm{Re}_\mu[\lambda_i(t)\oplus\lambda_j(t)] &= \frac{|1+\mu(t)(\lambda_i(t)\oplus\lambda_j(t))|-1}{\mu(t)}\\ &= \frac{|(1+\mu(t)\lambda_i(t))\|(1+\mu(t)\lambda_j(t))|-1}{\mu(t)}\\ &< \frac{|(1+\mu(t)\lambda_j(t))|-1}{\mu(t)}\\ &= \mathrm{Re}_\mu[\lambda_j(t)]\\ &\leqslant -\varepsilon\end{aligned}$$

for all $t\in\mathbb{T}$ and all $i,\ j=1,\dots,n$.

Therefore, $\mathrm{Re}_\mu[\lambda_i(t)\oplus\lambda_j(t)] < -\varepsilon<0$ for $0<\varepsilon<\frac{1}{\mu_{\max}}\leqslant\frac{1}{\mu(t)}$ and we also have the relationship $0<\varepsilon\leqslant|\mathrm{Re}_\mu[\lambda_i(t)\oplus\lambda_j(t)]|\leqslant|\lambda_i(t)\oplus\lambda_j(t)|$.

Thus

$$|\det[(A^{\mathrm{T}}(t)\otimes I)\oplus(I\otimes A^{\mathrm{T}}(t))]| = \left|\prod_{i,j=1}^{n}[\lambda_i(t)\oplus\lambda_j(t)]\right| \geqslant \varepsilon^{n^2}, \quad t\in\mathbb{T}. \tag{4.4}$$

Now it is clear that $(A^{\mathrm{T}}(t)\otimes I)\oplus(I\otimes A^{\mathrm{T}}(t))$ is invertible at each $t\in\mathbb{T}$ since the determinant in (4.4) is nonzero and bounded away from zero for all t. Since $A(t)$ and $\mu(t)$ are bounded above, $A^{\mathrm{T}}(t)\otimes I$ is bounded above, and hence the inverse

$$[(A^{\mathrm{T}}(t)\otimes I)\oplus(I\otimes A^{\mathrm{T}}(t))]^{-1}$$

is also bounded for all $t\in\mathbb{T}$. Since the right-hand side of (4.3) is constant, we conclude that $q(t)$ is bounded for all $t\in\mathbb{T}$ and hence there exists a positive constant ρ such that $Q(t)\leqslant\rho I$ for all $t\in\mathbb{T}$.

Clearly, $Q(t)\in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ and is symmetric. Now we show that there exists a $v>0$ such that

$$A^{\mathrm{T}}(t)Q(t)+(I+\mu(t)A^{\mathrm{T}}(t))Q(t)A(t)+(I+\mu(t)A(t))^{\mathrm{T}}Q^{\Delta}(t)(I+\mu(t)A(t))\leqslant -vI$$

for all $t\in\mathbb{T}$. Since $Q(t)$ satisfies (4.2), the above inequality is equivalent to

$$(I+\mu(t)A(t))^{\mathrm{T}}Q^{\Delta}(t)(I+\mu(t)A(t))\leqslant(1-v)I,$$

which gives

$$Q^{\Delta}(t)\leqslant(1-v)(I+\mu(t)A^{\mathrm{T}}(t))^{-1}(I+\mu(t)A(t))^{-1}. \tag{4.5}$$

Delta differentiating (4.2) with respect to t, we obtain

$$\begin{aligned}&A^{\mathrm{T}^\sigma}(t)Q^{\Delta}(t)+A^{\mathrm{T}^\Delta}(t)Q(t)+Q^{\Delta}(t)A^{\sigma}(t)+Q(t)A^{\Delta}(t)\\ &\quad+\mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t)+\mu^{\sigma}(t)A^{\mathrm{T}^\Delta}(t)Q(t)A(t)\\ &\quad+\mu^{\sigma}(t)A^{\mathrm{T}^\sigma}(t)Q^{\Delta}(t)A(t)+\mu^{\sigma}(t)A^{\mathrm{T}^\sigma}(t)Q^{\sigma}(t)A^{\Delta}(t)=0.\end{aligned}$$

Recalling $Q^{\sigma}(t) = \mu(t)Q^{\Delta}(t) + Q(t)$ the above becomes

$$\begin{aligned}
&A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + A^{\mathrm{T}^{\Delta}}(t)Q(t) + Q^{\Delta}(t)A^{\sigma}(t) + Q(t)A^{\Delta}(t)\\
&\quad + \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t)\\
&\quad + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A(t) + \mu(t)\mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A^{\Delta}(t)\\
&\quad + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + Q^{\Delta}(t)A^{\sigma}(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A(t) + \mu(t)\mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A^{\Delta}(t)\\
&\quad = -A^{\mathrm{T}^{\Delta}}(t)Q(t) - Q(t)A^{\Delta}(t) - \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t) - \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t)\\
&\qquad - \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t).
\end{aligned}$$

Transforming only the left-hand side, we have

$$\begin{aligned}
&A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + Q^{\Delta}(t)A^{\sigma}(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A(t) + \mu(t)\mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A^{\Delta}(t)\\
&\quad = A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + Q^{\Delta}(t)A^{\sigma}(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)(A(t) + \mu(t)A^{\Delta}(t))\\
&\quad = A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + Q^{\Delta}(t)A^{\sigma}(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A^{\sigma}(t).
\end{aligned}$$

Thus, we now have

$$\begin{aligned}
&A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t) + Q^{\Delta}(t)A^{\sigma}(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q^{\Delta}(t)A^{\sigma}(t)\\
&\quad = -A^{\mathrm{T}^{\Delta}}(t)Q(t) - Q(t)A^{\Delta}(t) - \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t) - \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t)\\
&\qquad - \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t).
\end{aligned} \tag{4.6}$$

For simplicity, let

$$\begin{aligned}
X = {}& A^{\mathrm{T}^{\Delta}}(t)Q(t) + Q(t)A^{\Delta}(t) + \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t)\\
&+ \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t).
\end{aligned}$$

Then the solution, $Q^{\Delta}(t)$, of the matrix equation (4.6) can be written as

$$Q^{\Delta}(t) = \int_{I^{\sigma}} e_{A^{\mathrm{T}^{\sigma}}(t)}(s,0)Xe_{A^{\sigma}(t)}(s,0)\Delta s, \quad t \in \mathbb{T}^{\kappa} = \mathbb{T},$$

where $I^{\sigma} := [0,\infty)_{\mathbb{S}^{\sigma}}$ and $\mathbb{S}^{\sigma} = \mu^{\sigma}(t)\mathbb{N}_0$. Now, to obtain a bound on $Q^{\Delta}(t)$, we use the boundedness of $Q(t)$, $Q^{\sigma}(t)$, $A(t)$, $A^{\Delta}(t)$, $\mu_{\max}$, and $\mu^{\Delta}_{\max}$. For any $n \times 1$ vector x and any t,

$$\begin{aligned}
&|x^{\mathrm{T}}e_{A^{\mathrm{T}^{\sigma}}(t)}(s,0)Xe_{A^{\sigma}(t)}(s,0)x|\\
&\quad = |x^{\mathrm{T}}e_{A^{\mathrm{T}^{\sigma}}(t)}(s,0)[A^{\mathrm{T}^{\Delta}}(t)Q(t) + Q(t)A^{\Delta}(t) + \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t)\\
&\qquad + \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t)]e_{A^{\sigma}(t)}(s,0)x|\\
&\quad \leqslant \|A^{\mathrm{T}^{\Delta}}(t)Q(t) + Q(t)A^{\Delta}(t) + \mu^{\Delta}(t)A^{\mathrm{T}}(t)Q(t)A(t)\\
&\qquad + \mu^{\sigma}(t)A^{\mathrm{T}^{\Delta}}(t)Q(t)A(t) + \mu^{\sigma}(t)A^{\mathrm{T}^{\sigma}}(t)Q(t)A^{\Delta}(t)\|x^{\mathrm{T}}e_{A^{\mathrm{T}^{\sigma}}(t)}(s,0)e_{A^{\sigma}(t)}(s,0)x.
\end{aligned}$$

Thus

$$\begin{aligned}
|x^{\mathrm{T}} Q^{\Delta}(t)x| &= \left| \int_{I^\sigma} x^{\mathrm{T}} e_{A^{\mathrm{T}\sigma}(t)}(s,0) X e_{A^\sigma(t)}(s,0)x \Delta s \right| \\
&\leqslant \| A^{\mathrm{T}^\Delta}(t)Q(t) + Q(t)A^\Delta(t) + \mu^\Delta(t)A^{\mathrm{T}}(t)Q(t)A(t) + \mu^\sigma(t)A^{\mathrm{T}^\Delta}(t)Q(t)A(t) \\
&\quad + \mu^\sigma(t)A^{\mathrm{T}\sigma}(t)Q(t)A^\Delta(t)\| x^{\mathrm{T}} Q^\sigma(t)x \\
&\leqslant (2\beta\|Q(t)\| + \mu^\Delta_{\max}\alpha^2\|Q(t)\| + 2\mu_{\max}\alpha\beta\|Q(t)\|)x^{\mathrm{T}} Q^\sigma(t)x \\
&= \|Q(t)\|(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max})x^{\mathrm{T}} Q^\sigma(t)x.
\end{aligned}$$

We now maximize the right-hand side over all x such that $\|x\| = 1$ to obtain

$$|x^{\mathrm{T}} Q^\Delta(t)x| \leqslant \|Q(t)\|\|Q^\sigma(t)\|(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max})$$

and after maximizing the left-hand side over all x such that $\|x\| = 1$ we have

$$\|Q^\Delta(t)\| \leqslant \rho^2(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max}), \quad t \in \mathbb{T}^\kappa.$$

Using α, $\mu_{\max}$, $\mu^\Delta_{\max}$, and the norm bound ρ on $Q(t)$ and $Q^\sigma(t)$, the bound β on $\|A^\Delta(t)\|$ can be chosen so that we can create a bound for $Q^\Delta(t)$ which in turn yields a value for v in (4.5).

Lastly, we show that there exists a positive constant η such that $\eta I \leqslant Q(t)$, for all $t \in \mathbb{T}$. For any t and any $n \times 1$ vector x,

$$\begin{aligned}
&[x^{\mathrm{T}} e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0)x]^{\Delta_s} \\
&\quad = x^{\mathrm{T}}[A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0) + e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0)A(t) \\
&\qquad + \mu(t)A^{\mathrm{T}}(t)e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0)A(t)]x \\
&\quad = x^{\mathrm{T}} e_{A^{\mathrm{T}}(t)}(s,0)[A^{\mathrm{T}}(t) + A(t) + \mu(t)A^{\mathrm{T}}(t)A(t)]e_{A(t)}(s,0)x \\
&\quad \geqslant (-2\alpha - \mu_{\max}\alpha^2)x^{\mathrm{T}} e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0)x.
\end{aligned}$$

As $s \to \infty$, we know that $e_{A(t)}(s,0) \to 0$, so that

$$-x^{\mathrm{T}} x = \int_I [x^{\mathrm{T}} e_{A^{\mathrm{T}}(t)}(s,0)e_{A(t)}(s,0)x]^{\Delta_s} \Delta s \geqslant (-2\alpha - \mu_{\max}\alpha^2)x^{\mathrm{T}} Q(t)x.$$

But of course this is equivalent to

$$Q(t) \geqslant \frac{1}{(2\alpha + \mu_{\max}\alpha^2)} I, \quad t \in \mathbb{T}.$$

So we set $\eta = \frac{1}{(2\alpha+\mu_{\max}\alpha^2)}$. $\square$

5. Perturbation results

It is also useful to consider state equations that are "close" to another linear state equation that is uniformly stable. In [19,20], as well as [25], if the stability of system (3.1) has already been determined by an appropriate Lyapunov function, then certain conditions on the perturbation matrix $F(t)$ guarantee stability of the perturbed linear system

$$z^{\Delta}(t) = [A(t) + F(t)]z(t). \tag{5.1}$$

Theorem 5.1. *Suppose the linear state equation* (3.1) *is uniformly stable. Then the perturbed linear dynamic equation* (5.1) *is uniformly stable if there exists some* $\beta \geqslant 0$ *such that for all* τ

$$\int_{\tau}^{\infty} \|F(s)\|\Delta s \leqslant \beta. \tag{5.2}$$

Proof. For any t_0 and $z(t_0) = z_0$, by Theorem A.6 the solution of (5.1) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s, \tag{5.3}$$

where $\Phi_A(t, t_0)$ is the transition matrix for system (3.1). By the uniform stability of (3.1), there exists a constant $\gamma > 0$ such that $\|\Phi_A(t, \tau)\| \leqslant \gamma$, for all $t, \tau \in \mathbb{T}$ with $t \geqslant \tau$. By taking the norms of both sides of (5.3), we have

$$\|z(t)\| \leqslant \gamma\|z_0\| + \int_{t_0}^{t} \gamma\|F(s)\| \|z(s)\|\Delta s, \quad t \geqslant t_0. \tag{5.4}$$

By Gronwall's Inequality in [5], a result in [10], and the inequality (5.2), we obtain

$$\begin{aligned}
\|z(t)\| &\leqslant \gamma\|z_0\|e_{\gamma\|F\|}(t, t_0) \\
&= \gamma\|z_0\| \exp\left(\int_{t_0}^{t} \frac{\mathrm{Log}(1 + \mu(s)\gamma\|F(s)\|)}{\mu(s)}\Delta s\right) \\
&\leqslant \gamma\|z_0\| \exp\left(\int_{t_0}^{\infty} \frac{\mathrm{Log}(1 + \mu(s)\gamma\|F(s)\|)}{\mu(s)}\Delta s\right) \\
&\leqslant \gamma\|z_0\| \exp\left(\int_{t_0}^{\infty} \gamma\|F(s)\|\Delta s\right) \\
&\leqslant \gamma\|z_0\|e^{\gamma\beta}, \quad t \geqslant t_0.
\end{aligned}$$

Since γ can be used for any t_0 and $z(t_0)$, the state equation (5.1) is uniformly stable. □

6. Instability criterion

We can also employ the unified timescale quadratic Lyapunov function to determine if system (3.1) is unstable. This is a very useful result in the case where the development of a suitable matrix $Q(t)$ is difficult and the possibility of an unstable system begins to arise. One type of instability criteria is developed in the next theorem.

Theorem 6.1. *Suppose there exists an* $n \times n$ *matrix* $Q(t) \in C^1_{rd}$ *that is symmetric for all* $t \in \mathbb{T}$ *and has the following two properties*:

(i) $\|Q(t)\| \leqslant \rho$,
(ii) $A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leqslant -vI$,
where $\rho, v > 0$. *Also suppose that there exists some* $t_a \in \mathbb{T}$ *such that* $Q(t_a)$ *is not positive semidefinite*. *Then the linear dynamic equation* (3.1) *is not uniformly stable*.

Proof. Suppose that $x(t)$ is the solution of (3.1) with initial conditions $t_0 = t_a$ and $x(t_0) = x(t_a) = x_a$ with $x_a^T Q(t_a)x_a < 0$. Then

$$\begin{aligned} x^T(t)Q(t)x(t) - x_0^T Q(t_0)x_0 &= \int_{t_0}^t [x^T(s)Q(s)x(s)]^{\Delta s}\Delta s \\ &\leqslant -v\int_{t_0}^t x^T(s)x(s)\Delta s \leqslant 0, \quad t \geqslant t_0. \end{aligned}$$

From this inequality,

$$x^T(t)Q(t)x(t) \leqslant x_0^T Q(t_0)x_0 < 0, \quad t \geqslant t_0.$$

With assumption (ii) we obtain

$$-\rho\|x(t)\|^2 \leqslant x^T(t)Q(t)x(t) \leqslant x^T(t_0)Q(t_0)x(t_0) < 0, \quad t \geqslant t_0,$$

which leads to

$$\|x(t)\|^2 \geqslant \frac{1}{\rho}|x^T(t)Q(t)x(t)| > 0, \quad t \geqslant t_0. \tag{6.1}$$

Again by employing assumption (ii),

$$\begin{aligned} v\int_{t_0}^t x^T(s)x(s)\Delta s &\leqslant x_0^T Q(t_0)x_0 - x^T(t)Q(t)x(t) \\ &\leqslant |x_0^T Q(t_0)x_0| + |x^T(t)Q(t)x(t)| \\ &\leqslant 2|x^T(t)Q(t)x(t)|, \quad t \geqslant t_0. \end{aligned}$$

Using (6.1), we finally obtain

$$\int_{t_0}^t x^T(s)x(s)\Delta s \leqslant \frac{2\rho}{v}\|x(t)\|^2, \quad t \geqslant t_0. \tag{6.2}$$

To end the proof, we show that $x(t)$ is unbounded. With an unbounded solution, we can conclude that (3.1) is not uniformly stable. Suppose there exists some $\gamma > 0$ so that $\|x(t)\| \leqslant \gamma$ for all $t \geqslant t_0$.

Then (6.2) implies

$$\int_{t_0}^{t} x^{\mathrm{T}}(s)x(s)\Delta s \leqslant \frac{2\rho\gamma^2}{\nu}, \quad t \geqslant t_0.$$

By this last inequality, $\|x(t)\| \to 0$ as $t \to \infty$, which contradicts (6.1). Thus, the solution $x(t)$ cannot be bounded, which shows that (3.1) is not uniformly stable. □

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Appendix A. A time scales primer

A.1. What are time scales?

A thorough introduction to dynamic equations on time scales is beyond the scope of this appendix. In short, the theory springs from the 1988 doctoral dissertation of Stefan Hilger [15] that resulted in his seminal paper [14] in 1990. These works aimed to unify and generalize various mathematical concepts from the theories of discrete and continuous dynamical systems. Afterwards, the body of knowledge concerning time scales advanced fairly quickly, culminating in the excellent introductory text in [5] and their more recent advanced monograph [4]. The material in this Appendix is drawn mainly from [5]. A succinct survey on time scales can be found in [2].

A *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. Thus time scales can be any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with continuous intervals. The majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

The *forward jump operator of* $\mathbb{T}$, $\sigma(t) : \mathbb{T} \to \mathbb{T}$, is given by $\sigma(t) = \inf_{s\in\mathbb{T}}\{s > t\}$. The *backward jump operator of* $\mathbb{T}$, $\rho(t) : \mathbb{T} \to \mathbb{T}$, is given by $\rho(t) = \sup_{s\in\mathbb{T}}\{s < t\}$. The *graininess function* $\mu(t) : \mathbb{T} \to [0, \infty)$ is given by $\mu(t) = \sigma(t) - t$. Here we adopt the conventions $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum element t), and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(t) = t$ if $\mathbb{T}$ has a minimum element t). For notational purposes, the intersection of a real interval $[a, b]$ with a time scale $\mathbb{T}$ is denoted by $[a, b] \cap \mathbb{T} := [a, b]_{\mathbb{T}}$.

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in \mathbb{T}$ is *left-scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$. If t is both left-scattered and right-scattered, we say t is *isolated*. If t is both left-dense and right-dense, we say t is *dense*. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then the composition $f(\sigma(t))$ is often denoted by $f^\sigma(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, define $f^\Delta(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|, \quad \forall s \in U.$$

The function $f^{\Delta}:\mathbb{T}^{\kappa}\to\mathbb{R}$ is called the *delta derivative* or the *Hilger derivative* of f on $\mathbb{T}^{\kappa}$. We say f is *delta differentiable* on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}^{\kappa}$.

The following theorem establishes several important observations regarding delta derivatives.

Theorem A.1. *Suppose $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^{\kappa}$.*

(i) *If f is delta differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is delta differentiable at t and $f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}$.*
(iii) *If t is right-dense, then f is delta differentiable at t if and only if $\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$ exists. In this case, $f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$.*
(iv) *If f is delta differentiable at t, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$.*

Note that f^{Δ} is precisely f' from the usual calculus when $\mathbb{T}=\mathbb{R}$. On the other hand, $f^{\Delta}=\Delta f=f(t+1)-f(t)$ (i.e. the forward difference operator) on the time scale $\mathbb{T}=\mathbb{Z}$. These are but two very special (and rather simple) examples of time scales. Moreover, the realms of differential equations and difference equations can now be viewed as but special, particular cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function.

A function $f:\mathbb{T}\to\mathbb{R}$ is *rd-continuous* if f is continuous at every right dense point $t\in\mathbb{T}$, and its left hand limit exists at each left dense point $t\in\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ will be denoted by $\mathrm{C_{rd}}=\mathrm{C_{rd}}(\mathbb{T})=\mathrm{C_{rd}}(\mathbb{T},\mathbb{R})$. A function $F:\mathbb{T}\to\mathbb{R}$ is called a (delta) *antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^{\kappa}$. The *Cauchy integral* or *definite integral* is given by $\int_a^b f(t)\Delta t=F(b)-F(a)$, for all $a,b\in\mathbb{T}$, where F is any (delta) antiderivative of f. Suppose that $\sup\mathbb{T}=\infty$. Then the *improper integral* is defined to by $\int_a^{\infty} f(t)\Delta t=\lim_{b\to\infty}F(t)|_a^b$ for all $a\in\mathbb{T}$. We remark that the delta integral can be defined in terms of a Lebesgue type integral [4] or a Riemann integral [5].

Theorem A.2 (*Existence of antiderivatives*).

(i) *Every rd-continuous function has an antiderivative. If $t_0\in\mathbb{T}$, then $F(t)=\int_{t_0}^{t} f(\tau)\,\Delta\tau$, $t\in\mathbb{T}$, is an antiderivative of f.*
(ii) *If $f\in\mathrm{C_{rd}}$ and $t\in\mathbb{T}^{\kappa}$, then $\int_t^{\sigma(t)} f(\tau)\,\Delta\tau=f(t)\mu(t)$.*
(iii) *Suppose $a,b\in\mathbb{T}$ and $f\in\mathrm{C_{rd}}$.*
(a) *If $\mathbb{T}=\mathbb{R}$, then $\int_a^b f(t)\,\Delta t=\int_a^b f(t)\,\mathrm{d}t$ (the usual Riemann integral).*
(b) *If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then*

$$\int_a^b f(t)\,\Delta t=\begin{cases}\sum_{t\in[a,b)_{\mathbb{T}}} f(t)\mu(t), & a<b,\\ 0, & a=b,\\ -\sum_{t\in[b,a)_{\mathbb{T}}} f(t)\mu(t), & a>b.\end{cases}$$

The last result above reveals that in the continuous case, $\mathbb{T}=\mathbb{R}$, definite integrals are the usual Riemann integrals from calculus. When $\mathbb{T}=\mathbb{Z}$, definite integrals correspond to definite sums from the difference calculus; see [21].

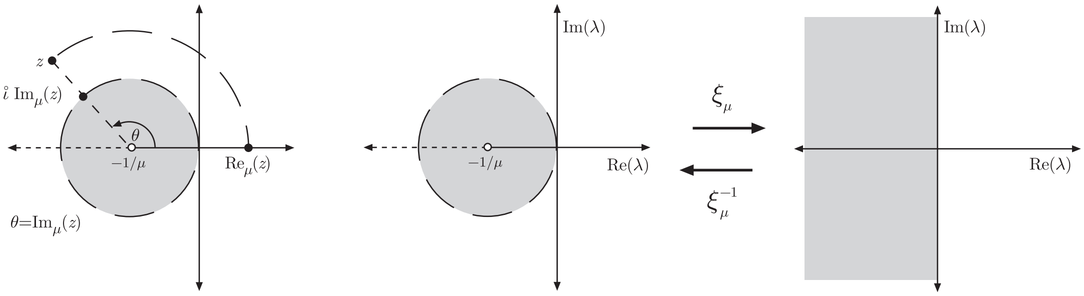

Fig. 1. Left: The Hilger complex plane. Right: The cylinder (A.1) and inverse cylinder (A.2) transformations map the familiar stability region in the continuous case to the interior of the Hilger circle in the general time scale case.

A.2. The Hilger complex plane

For $h>0$, define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis*, and the *Hilger imaginary circle* by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \quad \mathbb{R}_h := \left\{ z \in \mathbb{R} : z > -\frac{1}{h} \right\},$$
$$\mathbb{A}_h := \left\{ z \in \mathbb{R} : z < -\frac{1}{h} \right\}, \quad \mathbb{I}_h := \left\{ z \in \mathbb{C} : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},$$

respectively. For $h=0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{A}_0 := \emptyset$, and $\mathbb{I}_0 := i\mathbb{R}$. See Fig. 1.

Let $h>0$ and $z \in \mathbb{C}_h$. The *Hilger real part of z* is defined by $\mathrm{Re}_h(z) := \frac{|zh+1|-1}{h}$, and the *Hilger imaginary part of z* is defined by $\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh+1)}{h}$, where Arg(z) denotes the principal argument of z (i.e., $-\pi < \mathrm{Arg}(z) \leqslant \pi$). See Fig. 1.

For $h>0$, define the strip $\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leqslant \frac{\pi}{h}\}$, and for $h=0$, set $\mathbb{Z}_0 := \mathbb{C}$. Then we can define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h} \mathrm{Log}(1+zh), \quad h>0, \tag{A.1}$$

where Log is the principal logarithm function. When $h=0$, we define $\xi_0(z)=z$, for all $z \in \mathbb{C}$. It then follows that the *inverse cylinder transformation* $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ is given by

$$\xi_h^{-1}(z) = \frac{\mathrm{e}^{zh}-1}{h}. \tag{A.2}$$

See Fig. 1.

Since the graininess may not be constant for a given time scale, we will interchangeably subscript various quantities (such as ξ and ξ^{-1}) with $\mu=\mu(t)$ instead of h to reflect this.

A.3. Generalized exponential functions

The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, and this concept motivates the definition of the following sets:

$$\mathscr{R} = \{p : \mathbb{T} \to \mathbb{R} : p \in \mathrm{C_{rd}}(\mathbb{T}) \text{ and } 1 + \mu(t)p(t) \neq 0\ \forall t \in \mathbb{T}^\kappa\},$$
$$\mathscr{R}^+ = \{p \in \mathscr{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^\kappa\}.$$

The function $p : \mathbb{T} \to \mathbb{R}$ is *uniformly regressive* on $\mathbb{T}$ if there exists a positive constant δ such that $0 < \delta^{-1} \leqslant |1 + \mu(t)p(t)|,\ t \in \mathbb{T}^\kappa$. A matrix is regressive if and only if all of its eigenvalues are in $\mathscr{R}$. Equivalently, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A$ is invertible for all $t \in \mathbb{T}^\kappa$.

If $p \in \mathscr{R}$, then we define the *generalized time scale exponential function* by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for all } s, t \in \mathbb{T}.$$

The following theorem is a compilation of properties of $e_p(t, t_0)$ (some of which are counterintuitive) that we need in the main body of the paper.

Theorem A.3. *The function $e_p(t, t_0)$ has the following properties*:
(i) *If $p \in \mathscr{R}$, then $e_p(t,r)e_p(r,s) = e_p(t,s)$ for all $r, s, t \in \mathbb{T}$.*
(ii) *$e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$.*
(iii) *If $p \in \mathscr{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*
(iv) *If $1 + \mu(t)p(t) < 0$ for some $t \in \mathbb{T}^\kappa$, then $e_p(t,t_0)e_p(\sigma(t),t_0) < 0$.*
(v) *If $\mathbb{T} = \mathbb{R}$, then $e_p(t,s) = e^{\int_s^t p(\tau)\,\mathrm{d}\tau}$. Moreover, if p is constant, then $e_p(t,s) = e^{p(t-s)}$.*
(vi) *If $\mathbb{T} = \mathbb{Z}$, then $e_p(t,s) = \prod_{\tau=s}^{t-1}(1 + p(\tau))$. Moreover, if $\mathbb{T} = h\mathbb{Z}$, with $h > 0$ and p is constant, then $e_p(t,s) = (1 + hp)^{(t-s)/h}$*

If $p \in \mathscr{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then the dynamic equation

$$y^\Delta(t) = p(t)y(t) + f(t) \tag{A.3}$$

is called *regressive*.

Theorem A.4 (*Variation of constants*). *Let $t_0 \in \mathbb{T}$ and $y(t_0) = y_0 \in \mathbb{R}$. Then the regressive IVP* (A.3) *has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$ given by*

$$y(t) = y_0 e_p(t,t_0) + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

We say the $n \times 1$-vector-valued system

$$y^\Delta(t) = A(t)y(t) + f(t) \tag{A.4}$$

is *regressive* provided $A \in \mathscr{R}$ and $f : \mathbb{T} \to \mathbb{R}^n$ is a rd-continuous vector-valued function.

Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution to the IVP

$$Y^{\Delta}(t) = A(t)Y(t), \quad Y(t_0) = I_n, \tag{A.5}$$

where I_n is the $n \times n$-identity matrix, is called the *transition matrix* and it is denoted by $\Phi_A(t, t_0)$.

In this paper, we denote the solution to (A.5) as $\Phi_A(t, t_0)$ when $A(t)$ is time varying and denote the solution as $e_A(t, t_0) \equiv \Phi_A(t, t_0)$ (the *matrix exponential*, as in [5]) only when $A(t) \equiv A$ is a constant matrix. Also, if $A(t)$ is a function on $\mathbb{T}$ and the time scale matrix exponential function is a function on some other time scale $\mathbb{S}$, then $A(t)$ is constant with respect to $e_{A(t)}(\tau, s)$, for all $\tau, s \in \mathbb{S}$ and $t \in \mathbb{T}$. The following lemma lists some properties of the transition matrix.

Theorem A.5. *Suppose $A, B \in \mathcal{R}$ are matrix-valued functions on $\mathbb{T}$.*

(i) *Then the semigroup property $\Phi_A(t, r)\Phi_A(r, s) = \Phi_A(t, s)$ is satisfied for all $r, s, t \in \mathbb{T}$.*
(ii) $\Phi_A(\sigma(t), s) = (I + \mu(t)A(t))\Phi_A(t, s)$.
(iii) *If $\mathbb{T} = \mathbb{R}$ and A is constant, then $\Phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$.*
(iv) *If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, and A is constant, then $\Phi_A(t, s) = e_A(t, s) = (I + hA)^{(t-s)/h}$.*

We now present a theorem that guarantees a unique solution to the regressive $n \times 1$-vector-valued dynamic IVP (A.4).

Theorem A.6 (*Variation of constants*). *Let $t_0 \in \mathbb{T}$ and $y(t_0) = y_0 \in \mathbb{R}^n$. Then the regressive IVP* (A.4) *has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$ given by*

$$y(t) = \Phi_A(t, t_0)y_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{A.6}$$

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