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# A Multidimensional Extension of Papoulis' Generalized Sampling Expansion and Some of Its Applications

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## Abstract

The generalized sampling expansion (GSE), initially formulated by Papoulis, generalizes a broad class of extensions generated from the Shannon sampling theorem. A significant contribution of this expansion is that the merit of these different extensions can be compared under a common framework. In this paper, we will present two extensions of this generalization. First, an extension which considers processing of the samples generated under the framework of the GSE. The second extension is a multidimensional (M–D) extension of the GSE. The M–D extension of the GSE not only retains all of the merits of its 1–D counterpart but also results in fascinating geometric analyses which do not occur in its 1–D counterpart. This analysis is useful in a number of areas. Namely, the minimum sampling density of M–D bandlimited functions, the multiband sampling theorem and, while combining with the first extension, the multidensity digital signal processing which is also an M–D extension of the multirate digital signal processing.

## 1. Introduction

The generalized sampling expansion (GSE), initially formulated by Papoulis [1,2], eloquently generalizes a broad class of extensions generated from the Shannon sampling theorem [10]. Two well known extensions that fall under this generalization is the ordinate-slope sampling[1] [11,12] and the interlaced sampling[2] [4,11]. Given a $2\sigma$–bandlimited function $f(t)$ (the spectrum $F(\nu) = 0$ for $|\nu| \geq \sigma$), the Shannon sampling theorem allows the function to be represented by its samples $\{f(nT_n)\}$ where $T_n = 1/2\sigma$, the Nyquist interval.

[1] or sample–derivative sampling.

[2] or bunched sampling.

In a more general setting, the $L^{th}$ order GSE allows the function to be represented by $L$ sample sets: $\{\{g_i(nT_s)\} \,|\, 0 \leq i \leq L-1\}$, where every $g_i(t)$ is the output of a linear system, $h_i(t)$, given $f(t)$ is the common input:

$$g_i(t) = f(t) * h_i(t) \tag{1}$$

The sampling period, $T_s$, is $L$ times that of $T_n$. Clearly, the overall sampling period remains equal to the Nyquist interval. Both the ordinate–slope sampling and the interlaced sampling are the $L = 2$ cases of the GSE. The two linear systems for the ordinate–slope sampling are: $H_0(\nu)$ is an allpass filter and $H_1(\nu) = j2\pi\nu$. For interlaced sampling, $H_0(\nu)$ remains an allpass filter while $H_1(\nu)$ is changed to $\exp(j2\pi\alpha\nu)$ where $0 < \alpha < T_s$.

The GSE starts with a partitioning of the spectrum $F(\nu)$. In the $L^{th}$ order GSE, $F(\nu)$ is partitioned into $L$ equal portions as shown in figure 1. In particular, we let $\mathcal{A}$ be the region $(-\sigma, -\sigma + c)$ where $c = 2\sigma/L$. The ordering of the $L$ partitions is as follows:

$$\hat{F}_k(\nu) = F(\nu + jc) \qquad \nu \in \mathcal{A}\,;\, k = 0 \text{ to } L-1 \tag{2}$$

$F_k(\nu)$ is just the $k^{th}$ partition of $F(\nu)$ shifted over to $\mathcal{A}$.

Let $\hat{g}_i(t)$ be the sampling impules corresponding to the sample set $\{g_i(nT_s)\}$:

$$\hat{g}_i(t) = \sum_{n=-\infty}^{\infty} g_i(nT_s)\,\delta(t - nT_s)$$

Then for $\nu \in \mathcal{A}$:

$$\hat{G}_i(\nu) = \sum_{k=0}^{L-1} H_i(\nu + jc)\hat{F}_k(\nu) \qquad i = 0 \text{ to } L-1 \tag{3}$$

Equation (3) can be written into a matrix form:

$$\vec{G} = \underline{H}\vec{F} \tag{4}$$

With the knowledge of every partition, $\hat{F}_k(\nu)$, in $\mathcal{A}$, the signal can be restored by shifting every partition to it's original position. Shifting in the $\nu$ domain corresponds to modulation in $t$ domain. In particular,

$$\begin{aligned} f(t) &= \int_{-\sigma}^{\sigma} F(\nu) e^{j2\pi\nu t} d\nu \\ &= \int_{\nu \in \mathcal{A}} \sum_{k=0}^{L-1} \left[ \hat{F}_k(\nu) e^{j2\pi k c t} \right] e^{j2\pi\nu t} d\nu \quad (5) \end{aligned}$$

The square brackets encloses the modulation process required of shifting the partitions back to their original positions. Let $\vec{E}$ be the *carrier vector*:

$$\vec{E} = (\ 1 \quad e^{j2\pi ct} \quad e^{j2\pi 2ct} \quad \ldots \quad e^{j2\pi(L-1)ct}\ )^T \tag{6}$$

(the subscript $T$ denotes transposition). With the carrier vector, (5) can be written into the following form:

$$f(t) = \int_{\nu \in \mathcal{A}} \vec{E}^T \vec{F} e^{j2\pi\nu t} d\nu \tag{7}$$

By substituting (5) into (7), we obtain

$$f(t) = \int_{\nu \in \mathcal{A}} \vec{E}^T \underline{H}^{-1} \vec{G}\, e^{j2\pi\nu t} d\nu \tag{8}$$

The product $\vec{E}^T \underline{H}^{-1} \vec{G}$ in (8) generates the expression of the interpolation formula for restoring $f(t)$. In particular, let

$$\underline{H}^T \vec{Y} = \vec{E} \tag{9}$$

where

$$\vec{Y} = (\ Y_0(\nu,t) \quad Y_1(\nu,t) \quad \ldots \quad Y_{L-1}(\nu,t)\ )^T$$

By solving (9), we obtain the $L$ interpolation kernels:

$$y_i(t) = \frac{1}{c} \int_{\nu \in \mathcal{A}} Y_i(\nu,t) e^{j2\pi\nu t} d\nu \tag{10}$$

and the interpolation formula follows:

$$f(t) = \sum_{i=0}^{L-1} \sum_{n=-\infty}^{\infty} g_i(nT_s)\, y_i(t - nT_s) \tag{11}$$

Equations (9) to (11) are the core equations for the GSE. Restoring $f(t)$ from the $L$ sample sets assumes the invertibility of the matrix $\underline{H}$.

A significant contribution of the GSE is that the merit of different sampling extensions can be compared under a common framework. By utilizing the GSE, Cheung and Marks [3] demonstrated that there is a class of innocent–appealing yet ill-posed sampling theorems.

### 1.1 Digital Signal Processing

After $L$ sample sets are obtained and if $\underline{H}$ is invertible, we can restore $f(t)$. We may, however, want to process these samples in order for some other desirable outputs, say $f_p(t)$. For example we desire $f_p(t)$ to be a lowpass version of $f(t)$.

Let the processing be represented by a filter $H_p(\nu)$:

$$F_p(\nu) = H_p(\nu)\, F(\nu)$$

Using the same partitioning notation, we obtain

$$\hat{F}_{p,k}(\nu) = H_{p,k}(\nu) \hat{F}_k(\nu) \qquad \nu \in \mathcal{A} \tag{12}$$

where $F_{p,k}(\nu)$ is the $k^{th}$ of $F_p(\nu)$ shifted over to $\mathcal{A}$, or, in matrix form,

$$\vec{F}_p = \underline{H}_p \vec{F} \tag{13}$$

In this setting, $\underline{H}_p$ is a diagonal matrix. Without bothering with details, (9) is modified to become

$$\underline{H}^T \vec{Y} = \underline{H}_p\, \vec{E} \qquad \nu \in \mathcal{A} \tag{14}$$

Here,

$$\vec{Y} = (\ Y_{p,0}(\nu,t) \quad Y_{p,1}(\nu,t) \quad \ldots \quad Y_{p,L-1}(\nu,t)\ )^T$$

The procedure to obtain the interpolation kernels and the interpolation formula for obtaining $f_p(t)$ is outlined in (10) and (11).

The diagonal $\underline{H}_p$ easily represents various popular signaling processing methodologies. For lowpass filtering, only the elements in the center portion are nonzeros. For highpass filtering, elements at the central portion are zeroed. And for Hilbert transformation [4], the upper and the lower half of the entries are reversed in polarity.

In general, $\underline{H}_p$ need not be diagonal. In this setting, partitions of $F(\nu)$ are weighted and mixed together. If $\underline{H}_p$ is invertible, the scrambled signal $f_p(t)$ can be descrambled to obtain $f(t)$.

## 2. Multidimensional Extension

In the same spirit of its 1-D counterpart, the multidimensional (M–D) extension of GSE is formulated to consider sampling M–D bandlimited functions with the same general setting. Before we proceed to details, we will briefly present a summary of the M–D sampling theorem.

### Multidimensional Sampling Theorem

The M–D sampling theorem, initially presented by Peterson and Middleton [5], can be summarized as follows. Let $f(\vec{t})$ be a $N$–D ($N > 1$) $\mathcal{B}$-bandlimited function ($F(\vec{\nu}) = 0$ for $\vec{\nu} \notin \mathcal{B}$). The function is said *lowpass* bandlimited if $\mathcal{B}$ is a $N$–D hypersphere centered at the origin.

Let $f(\vec{t})$ be a $N$–D lowpass bandlimited function and $\underline{V}$ be the sampling matrix. Then sampling $f(\vec{t})$ produces the following impulses:

$$\hat{f}(\vec{t}) = \sum_{\vec{n}} f(\underline{V}\vec{b})\delta(\vec{t} - \underline{V}\vec{n})$$

where

$$\sum_{\vec{n}} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty}$$

and $\delta(\vec{t})$ is the $N$–D Dirac delta function. Given $D = 1/|\underline{V}|$ the sampling density and $\underline{U}$ the replication matrix, where $\underline{U} = \left[\underline{V}^T\right]^{-1}$, the spectrum of $\hat{f}(\vec{t})$ is a periodic replications of $F(\vec{\nu})$:

$$\hat{F}(\vec{\nu}) = D \sum_{\vec{m}} F(\vec{\nu} - \underline{U}\vec{m})$$

Let $\mathcal{C}$ denotes a period of the replications $\hat{F}(\vec{\nu})$. For convenience, $\mathcal{C}$ is referred as a "cell". Restoration of $f(\vec{t})$ from $\hat{f}(\vec{t})$ is *via* interpolation:

$$f(\vec{t}) = \sum_{\vec{n}} f(\underline{V}\vec{n})h(\vec{t} - \underline{V}\vec{n})$$

The interpolation kernel is evaluated by

$$h(\vec{t}) = \frac{1}{D}\int_{\mathcal{C}_o} e^{j2\pi\vec{\nu}^T\vec{t}}d\vec{\nu}$$

where $\mathcal{C}_o$ is a cell containing only the zeroth order replication.

## 2.1 Multidimensional GSE

In identical to its 1–D counterpart, the M–D GSE allows $f(\vec{t})$ to be represented by $L$ sample sets $\{\{g_i(\underline{V}_d\vec{n})\} \,|\, i = 0 \text{ to } L-1\}$ where $g_i(\vec{t})$ is the output of a linear system, $h_i(\vec{t})$, given $f(\vec{t})$ the input. The matrix $\underline{V}_d$ is the sampling matrix of these $L$ sample sets. Here, we let $\underline{U}_d$ be the corresponding replication matrix and $\mathcal{C}_d$ be a replication period, which will be referred as "subcell" for convenience.

Again, the M–D GSE starts with partitioning the spectrum $F(\nu)$. In analogous to the 1–D GSE, the spectrum is partitioned into $L$ identical partitions. An example of the partitioning is shown in figure 2. The larger square is a cell $\mathcal{C}$ and the smaller square is a subcell, which is denoted by $\mathcal{C}_d$. Let $\mathcal{C}_{do}$ be the reference subcell. A method of locating $\mathcal{C}_{do}$ is outlined in [6]. Then

$$\hat{G}_i(\vec{\nu}) = \sum_{\ell=0}^{L-1} H_i(\vec{\nu}+\underline{U}_d\vec{k}_\ell)\hat{F}_\ell(\vec{\nu}) \qquad i = 0 \text{ to } L-1$$

where $\hat{F}_\ell(\vec{\nu})$ is the $\ell^{th}$ partition of $F(\vec{\nu})$ shifted over to $\mathcal{C}_{do}$, and $\vec{k}_\ell$ is an integer L-tuplet which represents the integer $\ell$. The procedure to formulate the M–D GSE is identical to that of the 1–D. Without going into details, restoring $f(\vec{t})$ from the $L$ sample sets is *via* $N$–D interpolation:

$$f(\vec{t}) = \sum_{i=0}^{L-1}\sum_{\vec{n}} g_i(\underline{V}_d\vec{n})y_i(\vec{t} - \underline{V}_d\vec{n}) \qquad (15)$$

The interpolation kernels is evaluated from

$$y_i(\vec{t}) = \frac{1}{D}\int_{\vec{\nu}\in\mathcal{C}_{do}} Y_i(\vec{\nu},\vec{t})e^{j2\pi\vec{\nu}^T\vec{t}}d\vec{\nu} \quad i = 0 \text{ to } L-1 \qquad (16)$$

The expression of the interpolation kernels, $Y_i(\vec{\nu},\vec{t})$'s, are solved from the same matrix equation in (9). The vector $\vec{E}$ in this M–D setting is:

$$\vec{E} = \begin{pmatrix} e^{j2\pi\vec{\nu}^T\underline{V}\vec{k}_0} & e^{j2\pi\vec{\nu}^T\underline{V}\vec{k}_1} & \cdots & e^{j2\pi\vec{\nu}^T\underline{V}\vec{k}_{L-1}} \end{pmatrix}^T$$

The multidimensional extension of GSE not only retains all the merits in its 1–D setting but also results in fascinating geometric analyses which do not present in its 1–D counterpart. These new results can immediately be applied to a number of areas.

## 2.2 Sampling Density Reduction

Under the condition that the spectral replications $\hat{F}(\vec{\nu})$ contain gaps (i.e. regions where $\hat{F}(\vec{\nu})$ is identically zero), the samples are shown to be linearly dependent [7,13]. By definition, a subset of samples can be deleted and restored by those remaining.

Due to the more complicated geometry of M–D bandlimited functions, gaps exits even when the function is sampled at the Nyquist density [13]. Using our running example again. Sampling the 2–D circularly bandlimited function at the Nyquist density corresponding to a hexagonal replication geometry. Clearly, the gaps exist among the replications.

By utilizing the M–D GSE, samples can be deleted periodically or decimated as long as gaps exists among the spectral replications $\hat{F}(\vec{\nu})$ [6,7]. Since the deletion is periodic, the overall sampling density is reduced. The geometric analyses in the M–D GSE shows that if $q$ subcells are subsumed within the gaps of a cell, the up to $q$ sample sets can be excluded from being used to restore $f(\vec{t})$. With reference to figure 2, we have four subcells subsumed with the gaps, therefore up to four sample sets can be excluded.

In particular, let $\mathcal{M}$ be the index set corresponding to the $q$ sample sets being excluded. The matrix equation in (9) is reduced to have a dimension of $L - q$:

$$\underline{\tilde{H}}\tilde{Y} = \tilde{E} \qquad (17)$$

and the resulting interpolation formula to restore $f(\vec{t})$ in (15) is also reduced:

$$f(\vec{t}) = \sum_{i \notin \mathcal{M}} \sum_{\vec{n}} g_i(\underline{V}_d\vec{n}) y_i(\vec{t} - \underline{V}_d\vec{n}) \qquad (18)$$

Hence, as long as there are $q$ subcells subsumed within the gaps of a cell, up to $q$ sample sets can be excluded in the restoration of $f(\vec{t})$. The overall sampling density is reduced by a factor of $q/L$.[3] Of course, the existence of solution is dependent on the invertibility of $\tilde{\underline{H}}$. Of all the $\binom{q}{L}$ combinations, we can show that there exists at least one combination which has well-posed solution [6].

In general, the smaller the subcells, the higher the $q$'s and $L$'s. We found that the ultimate reduction ratio yields the minimum density which is equal to the support of $F(\vec{\nu})$. This result certainly is valid for the 1-D case.

## 2.3 Multiband Sampling Theorem

The result in the last section can be applied directly to sample bandpass or in general multiband [8] functions, both in 1-D and M-D. This is because this class of functions contains gaps in their spectrum and therefore in their spectral replications as well.

By applying the result to multiband functions, we can sample a bandpass or multiband function directly at or arbitrarily close to the minimum density. We, however, expect that the posedness of this class of sampling theorems adverse with the degree of sampling jitters.

## 2.4 Multidensity Digital Signal Processing

The sampling deletion notation discussed in the last section fits the description of $q/L$ sampling rate conversion in the scenario of the multirate digital signal processing (MR-DSP) [9]. The MD-GSE may be utilized for this particular purpose.

By extending the MR-DSP to higher dimension, we may refer this extension as multidensity digital signal processing (MD-DSP), the M-D GSE can also be used to form the base of analysis for this particular extension. With reference to the formulation of the MR-DSP, the $M/L$ sampling density (rate) conversion can be implemented *via* two stages: a $M^{th}$ order decimation cascaded by a $L^{th}$ order interpolation. For the $M^{th}$ order decimation, the samples $\{f(\underline{V}\vec{n})\}$ first of all is subdivided into M groups: $\{\{f_r(\underline{V}_d\vec{n})\} \,|\, r = 0 \text{ to } M-1\}$ where

$$f_r(\underline{V}_d\vec{n}) = f(\underline{V}_d\vec{n} + \underline{V}\vec{k}_r)$$

where $\vec{k}_r$ is a $N$-tuplet representing the integer $r$. Only one sample set is retained and the other $M-1$ sets is discarded. For the $L^{th}$ order interpolation, only one sample set is nonzero and we need to generate the other $L-1$ sample sets. The discrete interpolation is usually implemented by a lowpass filter.

By utilizing the M-D GSE into MD-DSP, we obtain an alternate approach. The difference in using the two approaches is that the MR-DSP approach always provides uniform sampling geometries after decimation or interpolation, while the M-D GSE in general yields recurrent periodic sampling geometries [6,7,14]. Nevertheless, the M-D GSE approach provides geometric insights in the analysis as well as freedom to choose the sample sets be excluded in decimation or be restored in interpolation. Both the methodologies of decimation as well as the formulation of interpolation is presented in sections 2.2 and 2.3.

By coupling the DSP formulation presented in section 1.1, the M-D GSE can be furtherly be used to design interpolation and decimation filters used in the scenario of the MD-DSP.

# 3. Conclusion

In this paper, we consider the M-D extension of the GSE. The resulting geometric analyses allow us to apply the M-D extension to a number of application areas. Specifically, we can apply this extension to sampling density reduction, bandpass or multiband sampling theorems. The M-D GSE can also be applied to extend the multirate digital signal processing to higher dimensions.

[3] We refer this as $q^{th}$ order sampling reduction

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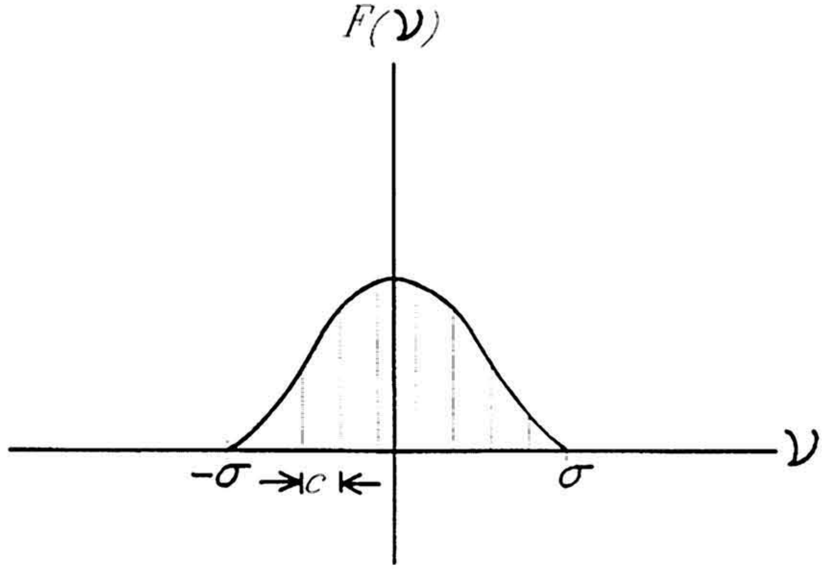


**Figure 1** The partitioning of the baseband of a 1-D $2\sigma$-bandlimited function in the generalized sampling expansion.

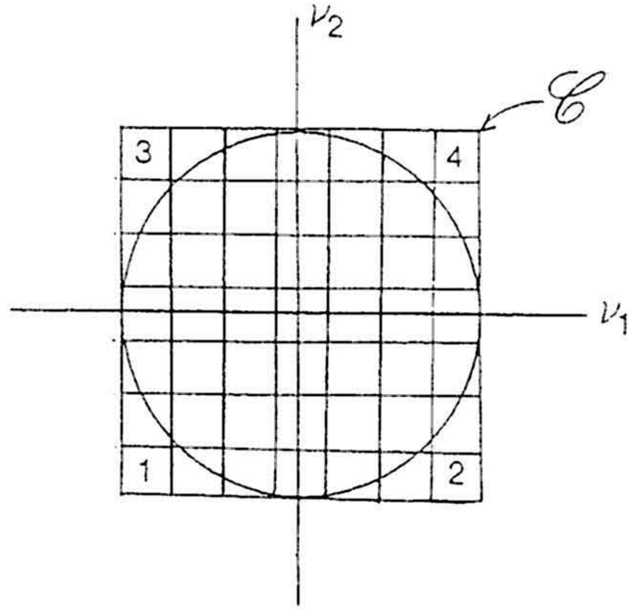


**Figure 2** An example of a partitioning of a 2-D cell in the multidimensional generalized sampling expansion.