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Noise sensitivity of interpolation and extrapolation matrices

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The noise sensitivity of interpolation and extrapolation matrices is investigated. For certain bandwidth and truncation parameters, the interpolation matrix is shown to yield results at a lower noise level than the input data. The input noise level, however, can be lowered by filtering the result. The noise level in the interpolated interval is shown to be lower near where the image is known. The extrapolation matrix is shown to be ill-conditioned, thus demonstrating severe sensitivity to input noise.

I. Introduction

Various closed-form discrete extrapolation algorithms have been proposed.[1–5] With a slight modification, these algorithms can also be applied to interpolation problems.

In this paper, we numerically explore noise sensitivity of both the interpolation and extrapolation matrices. Both are shown to be less sensitive near where the object is known. The interpolation matrix is shown to perform well, while the extrapolation matrix is ill-conditioned.

II. Preliminaries

Define the bandlimiting operator $\mathcal{B}_W$ by

$$\mathcal{B}_W s(x) = s(x) * 2W \operatorname{sinc} 2Wx, \tag{1}$$

where $\operatorname{sinc}\xi = \sin(\pi\xi)/(\pi\xi)$. We say that $f(x)$ is bandlimited with a bandwidth $\leq 2W$ if

$$\mathcal{B}_W f(x) = f(x). \tag{2}$$

Consider the object degradations

$$g_i(x) = f(x)\,[1 - \operatorname{rect}(x/T)], \tag{3a}$$

$$g_e(x) = f(x) \operatorname{rect}(x/T), \tag{3b}$$

where T is a duration, and

$$\operatorname{rect}(x) = \begin{cases} 1; & |x| \leq 1/2 \\ 0; & |x| > 1/2. \end{cases}$$

Regaining $f(\cdot)$ from $g_e(\cdot)$ is extrapolation and from $g_i(\cdot)$ is interpolation. The former is an ill-posed problem in the sense that a small amount of noise on $g_e(\cdot)$ can cause large error.[6–9] Interpolation, on the other hand, is well-posed.[6,9] The ratio of output to input error energy can be bound.

Note that, from Eqs. (2) and (3a), we can write $g_i(x) = [1 - \operatorname{rect}(x/T)\mathcal{B}_W]f(x)$. Inversion of the operator in square brackets would result in closed-form interpolation. If the problem is treated digitally, the sample values of $g_i(\cdot)$ and $f(\cdot)$ are placed in the vectors $\mathbf{g}_i$ and $\mathbf{f}$. The $\mathcal{B}_W$ operator becomes a low-pass matrix B_W which, for example, can be formed as in Ref. 10. The rect function is replaced by a square matrix, R_T, with 1s and 0s placed appropriately along the diagonal and zero elsewhere. Then Eq. (3a) can be written as $\mathbf{g}_i = [\mathrm{I} - \mathrm{R}_T\mathrm{B}_W]\mathbf{f}$. Inverting gives

$$\mathbf{f} = \mathcal{I}\mathbf{g}_i, \tag{4}$$

where we shall refer to

$$\mathcal{I} = [\mathrm{I} - \mathrm{R}_T\mathrm{B}_W]^{-1} \tag{5}$$

as the interpolation matrix

A parallel development can be applied to extrapolation. Specifically, $g_e(x) = \{1 - [1 - \operatorname{rect}(x/T)]\mathcal{B}_W\}f(x)$. Thus,

$$\mathbf{f} = \mathcal{E}\mathbf{g}_e, \tag{6}$$

where

$$\mathcal{E} = [\mathrm{I} - (\mathrm{I} - \mathrm{R}_T)B_W]^{-1} \tag{7}$$

is the extrapolation matrix.[1]

III. Matrix Structure

The analysis of the structure of the interpolation and extrapolation matrices is in order. Since the input vector is zero over the interval to be interpolated, a portion of the interpolation matrix picture in Fig. 1 is not used. This portion is shown shaded and is titled Don't Care. Furthermore, the output vector must be

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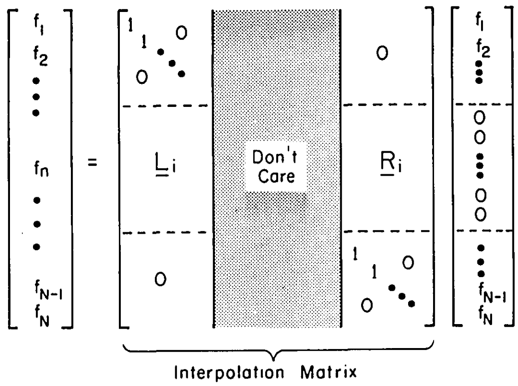

Fig. 1. Structure of the interpolation matrix $\mathcal{J}$.

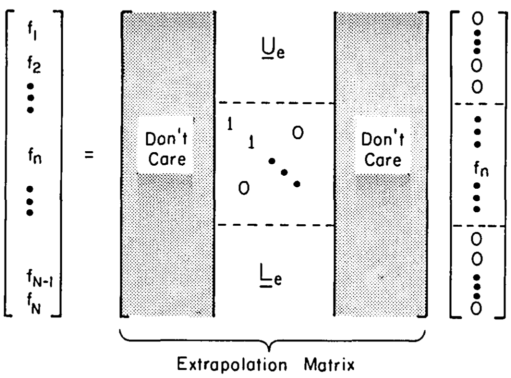

Fig. 2. Structure of the extrapolation matrix $\mathcal{E}$.

equal to the input vector at those points where the input vector is nonzero. The interpolation matrix performs this operation by mimicking an identity matrix in the four corners as shown in Fig. 1. The submatrices L_i and R_i thus completely suffice to specify $\mathcal{J}$. As the problem is presented, the interpolation matrix is symmetric. It is not if the time and frequency intervals are asymmetric.

A similar structure describes the extrapolation matrix. As is shown in Fig. 2, there are now two Don't Care strips. The subidentity matrix is centered. As shown, the matrix is solely determined by the submatrices U_e and L_e.

IV. Noise Sensitivity

The restoration schemes in both Eqs. (4) and (6) are linear. Thus, for the case of additive noise, the corresponding output noise is also additive and signal independent.

Let $\xi(x)$ denote a zero mean stationary random process so that the corresponding sample vector $\boldsymbol{\xi}$ consists of statistically independent identically distributed elements, each with variance σ^2. Such would be the case, for example, if $\xi(x)$ were a Gaussian random process with sufficiently short duration autocorrelation. Denote the corresponding interpolation and extrapolation input noise by, respectively,

$$\xi_i(x) = [1 - \mathrm{rect}(x/T)]\xi(x),$$

$$\xi_e(x) = \mathrm{rect}(x/T)\xi(x).$$

If $\mathbf{g}_i + \boldsymbol{\xi}_i$ is used in lieu of $\mathbf{g}_i$ in Eq. (4), the corresponding result is $\mathbf{f} + \boldsymbol{\eta}_i$, where

$$\boldsymbol{\eta}_i = \mathcal{J}\boldsymbol{\xi}_i. \tag{8}$$

Similarly, the error for extrapolation is $\boldsymbol{\eta}_e = \mathcal{E}\boldsymbol{\xi}_e$. If $\boldsymbol{\xi}_i$ is zero mean, so is $\boldsymbol{\eta}_i$. That is, $E[\boldsymbol{\eta}_i] = \mathcal{J}E[\boldsymbol{\xi}_i] = \mathbf{0}$, where E denotes the expected value operator, and $\mathbf{0}$ is the zero vector. Similarly, $E[\boldsymbol{\xi}_e] = \mathbf{0} \rightarrow E[\boldsymbol{\eta}_e] = \mathbf{0}$.

Of greater interest is the output's variance, which is a measure of the noise sensitivity of the restoration algorithm. With attention to the interpolation matrix structure shown in Fig. 1, we can write Eq. (8) as

$$(\eta_i)_m = \sum_{n \notin \text{Don't Care}} i_{mn}(\xi_i)_n,$$

where $(\eta_i)_m$ and $(\xi_i)_m$ are the mth elements of $\boldsymbol{\eta}_i$ and $\boldsymbol{\xi}_i$, and i_{mn} denotes the mnth elements of $\mathcal{J}$. Since the input vector is zero over an interval, the summation over the Don't Care portion of the matrix is excluded in the sum.

Since $\mathrm{var}(\xi_i)_n = \sigma^2$ for all n and all $(\xi_i)_n$ are statistically independent, it follows that[11]

$$\mathrm{var}(\eta_i)_m = \sigma^2 \sum_{n \notin \text{Don't Care}} i_{mn}^2$$

or, in normalized form,

$$(\Sigma_i)_m^2 \equiv \frac{\mathrm{var}(\eta_i)_m}{\sigma^2} = \sum_{n \notin \text{Don't Care}} i_{mn}^2.$$

The normalized variance of mth element of $\boldsymbol{\eta}_i$ can thus be obtained by summing the squares of the elements in the mth row of $\mathcal{J}$ outside the Don't Care region. As expected, the variance of the output where the signal is known is the same as the input. It is the variance of the elements of $\boldsymbol{\eta}_i$ corresponding to zero element inputs which is of interest.

A similar analysis applied to the extrapolation matrix in Fig. 2 yields

$$(\Sigma_e)_m^2 = \frac{\mathrm{var}(\eta_e)_m}{\sigma^2} = \sum_{n \notin \text{Don't Care}}^{n} e_{mn}^2,$$

where e_{mn} is the mnth element of $\mathcal{E}$, etc.

Once interpolation or extrapolation is performed, we may wish to low-pass filter the result to rid ourselves of high-frequency noise components. Equivalently, we can define revised interpolation and extrapolation matrices by $\hat{\mathcal{J}} = \mathrm{B}_W\,\mathcal{J}$ and $\hat{\mathcal{E}} = \mathrm{B}_W\,\mathcal{E}$. The corresponding revised normalized output variances are

$$(\hat{\Sigma}_i)_m^2 = \sum_{n \notin \text{Don't Care}}^{n} \hat{i}_{mn}^2,$$

$$(\hat{\Sigma}_e)_m^2 = \sum_{n \notin \text{Don't Care}} \hat{e}_{mn}^2,$$

where $\hat{i}_{mn}$ and $\hat{e}_{mn}$ are the mnth elements of $\hat{\mathcal{J}}$ and $\hat{\mathcal{E}}$, respectively.

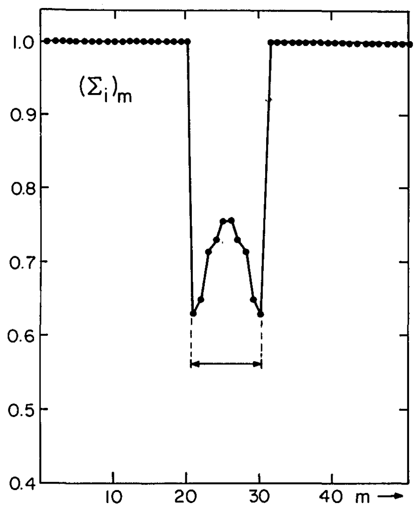

Fig. 3. Normalized noise level for interpolation, $\beta = 3$. The standard deviation for the ten interpolated points (within arrow) is lower than that of the data.

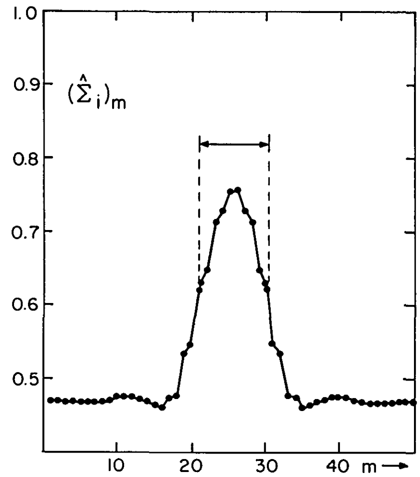

Fig. 4. Normalized noise level of Fig. 3 after high-frequency noise components have been filtered.

V. Numerical Results

For the examples to follow, we use Sabri and Steenaart's low-pass matrix.[10] The matrix is parametrized by the filter cutoff frequency β (the digital equivalent of W) expressed as the number of sample points in the frequency domain. All vectors are of length 50. For interpolation, ten centered elements corresponding to the T interval are filled from knowledge of the tails. For extrapolation, the ten centered elements are used to approximate the tails. Additive noise is white and Gaussian, with zero mean and a standard deviation of $\sigma = 0.1$.

Example 1: For interpolation with $\beta = 3$, the normalized output standard deviation $(\sum_i)_m$ is shown in Fig. 3. Interestingly, the SNR for the interpolated portion is superior to the data. This bothersome observation is resolved when we recall the high-frequency noise has not yet been smoothed. The revised normalized standard deviation $(\hat{\sum}_i)_m$ reveals that, after filtering, the noise level is reduced by a factor of 2 (see Fig. 4).

Example interpolation results are shown in Fig. 5. The original sinc signal is shown in Fig. 5(a). Using the tails as data, the interpolation matrix restoration of the ten inner data points is shown in Fig. 5(b). The result is graphically indistinguishable from the original but does peak at 1.04 instead of 1.00 due to truncation and quantization error.

Figure 5(c) shows interpolation when additive noise is added to the tail data. As predicted by Fig. 3, the interpolation is smoother than the data, peaking at 1.13. The result of filtering is shown in Fig. 5(d). The tails are smoothed and the interpolation remains unchanged.

Example 2: The interpolated noise level is not always an improvement of the data noise level as witnessed by the $\beta = 5$ plot for $(\sum_i)_m$ shown in Fig. 6. Since the bandwidth is larger, the omitted data can have more

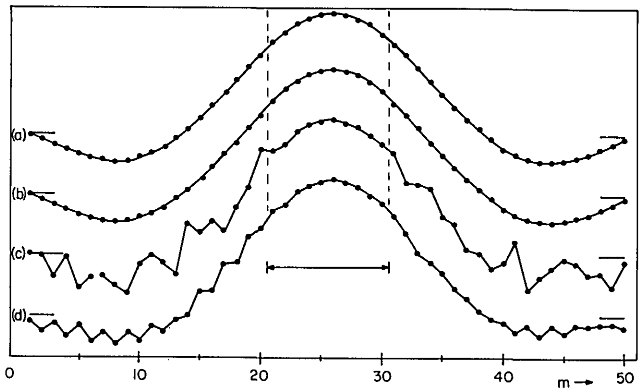

Fig. 5. Interpolation of ten points of a sinc, $\beta = 3$: (a) object; (b) noiseless interpolation; (c) interpolation in the presence of additive noise; and (d) filtered result.

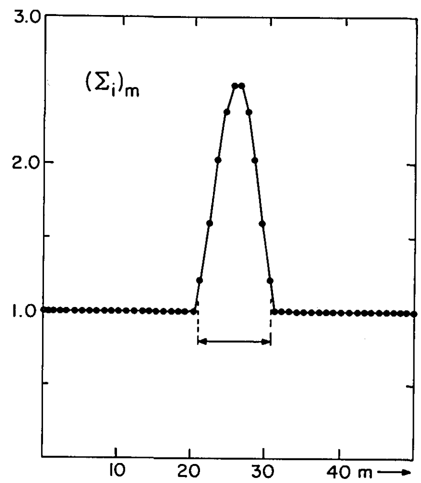

Fig. 6. Normalized noise level for interpolation, $\beta = 5$.

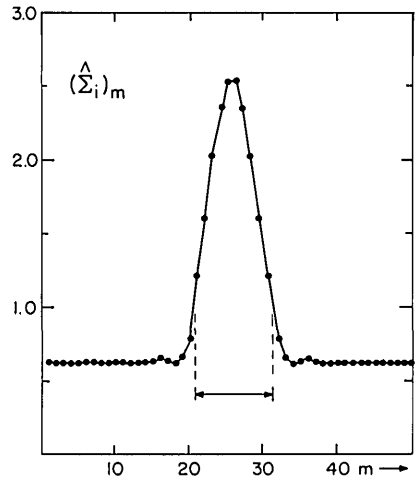

Fig. 7. Normalized noise level of Fig. 6 after high-frequency noise components have been filtered.

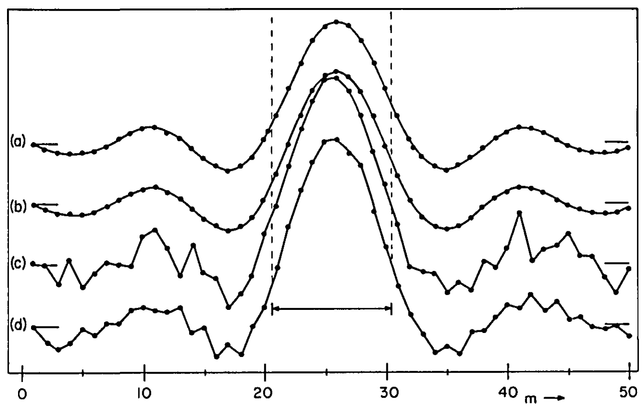

Fig. 8. Interpolation of ten points of a sinc, $\beta = 5$: (a) object; (b) noiseless interpolation; (c) interpolation in the presence of additive noise; and (d) filtered results.

structure within the interpolation interval. We would thus expect greater noise sensitivity. The normalized output variance after filtering is shown in Fig. 7. As before, the noise level in the tails is reduced.

An example of interpolation with this matrix is shown in Fig. 8. The original sinc object is shown in Fig. 8(a) and the corresponding interpolation in Fig. 8(b). For comparison purposes, the peak values are, respectively, unity and 1.09. Noise was added to the tails. The corresponding interpolation shown in Fig. 8(c) is smoother in the interpolation region but, in accordance with our prediction of greater noise sensitivity, has a peak value of 1.52. This same peak value occurs in Fig. 5(d), where the result has been filtered.

Example 3: For extrapolation with $\beta = 3$, the normalized standard deviation curves are illustrated in Fig. 9. The two curves are indistinguishable outside the interval where the data are known. Note that the vertical scale is logarithmic. Clearly, small levels of noise

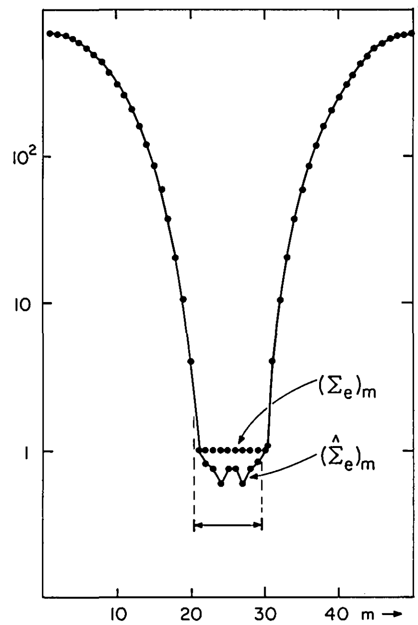

Fig. 9. Normalized and filtered noise level for the extrapolation matrix, $\beta = 3$.

on the input will yield enormous output values. This is in addition to error due to sampling.

Matrices demonstrating such noise sensitivity properties are labeled ill-conditioned.[12] Examples of application of extrapolation matrices in two dimensions, including effects of noise, can be found in Ref. 13.

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