Fuzzy Complement, t-norms & co-norms, from Klir & Yuan

An example of a fuzzy complement that is continuous (Axiom c3) but not involutive (Axiom c4) is the function

$$c(a) = \frac{1}{2}(1 + \cos \pi a),$$

which is illustrated in Fig. 3.3b. The failure of this function to satisfy the property of involution can be seen by noting, for example, that $c(.33) = .75$ but $c(.75) = .15 \neq .33$.

One class of involutive fuzzy complements is the *Sugeno class* defined by

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a}, \tag{3.5}$$

where $\lambda \in (-1, \infty)$. For each value of the parameter λ, we obtain one particular involutive fuzzy complement. This class is illustrated in Fig. 3.4a for several different values of λ. Note

that the shape of the function is affected as the value of λ is changed. For $\lambda = 0$, the function becomes the classical fuzzy complement defined by (3.1).

Another example of a class of involutive fuzzy complements is defined by

$$c_w(a) = (1 - a^w)^{1/w}, \tag{3.6}$$

where $w \in (0, \infty)$; let us refer to it as the *Yager class* of fuzzy complements. Figure 3.4b illustrates this class of functions for various values of w. Here again, changing the value of the parameter w results in a deformation of the shape of the function. When $w = 1$, this function becomes the classical fuzzy complement of $c(a) = 1 - a$.

Several important properties are shared by all fuzzy complements. These concern the *equilibrium* of a fuzzy complement c, which is defined as any value a for which $c(a) = a$. In other words, the equilibrium of a complement c is that degree of membership in a fuzzy set A which equals the degree of membership in the complement cA. For instance, the equilibrium value for the classical fuzzy complement given by (2.1) is .5, which is the solution of the equation $1 - a = a$.

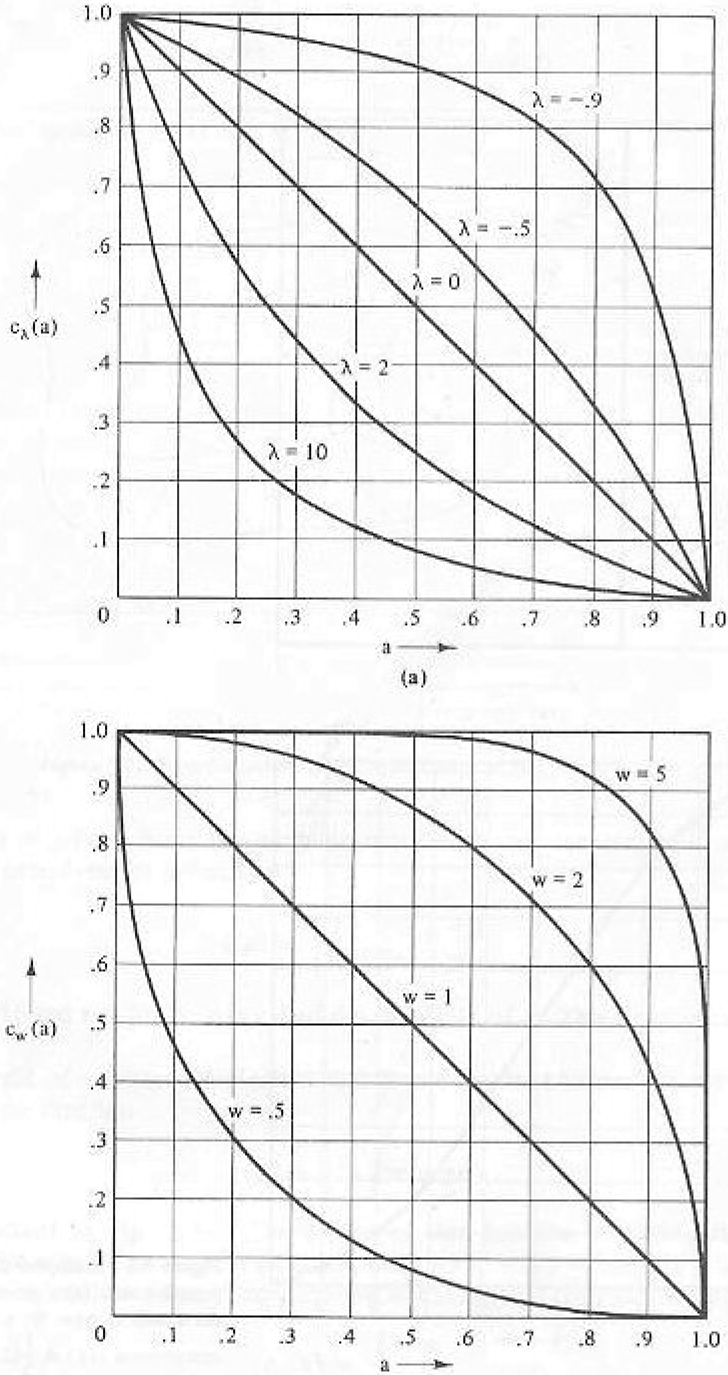

Figure 3.4 Examples from two classes of involutive fuzzy complements: (a) Sugeno class; (b) Yager class.

TABLE 3.2 SOME CLASSES OF FUZZY INTERSECTIONS (t-NORMS)

Reference	Formula $i(a,b)$	Decreasing generator $f(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a}-1\right)^{\lambda}$	$\lambda > 0$	$i_{\min}(a,b)$	$\frac{ab}{a+b-ab}$ when $\lambda = 1$	$\min(a,b)$
Frank [1979]	$\log_s\left[1+\frac{(s^a-1)(s^b-1)}{s-1}\right]$	$-\ln\left(\frac{s^a-1}{s-1}\right)$	$s > 0, s \neq 1$	$\min(a,b)$	ab as $s \to 1$	$\max(0, a+b-1)$
Hamacher [1978]	$\frac{ab}{r+(1-r)(a+b-ab)}$	$-\ln\left(\frac{a}{r+(1-r)a}\right)$	$r > 0$	$\frac{ab}{a+b-ab}$	ab when $r = 1$	$i_{\min}(a,b)$
Schweizer & Sklar 1 [1963]	$[\max(0, a^p+b^p-1)]^{\frac{1}{p}}$	$1-a^p$	$p \neq 0$	ab	$\max(0, a+b-1)$, when $p = 1$; $\frac{ab}{a+b-ab}$, when $p = -1$.	$i_{\min}(a,b)$ as $p \to \infty$; $\min(a,b)$ as $p \to -\infty$.
Schweizer & Sklar 2	$1-[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}$	$\ln[1-(1-a)^p]^{\frac{1}{p}}$	$p > 0$	$i_{\min}(a,b)$	ab when $p = 1$	$\min(a,b)$
Schweizer & Sklar 3	$\exp(-(\lvert\ln a\rvert^p+\lvert\ln b\rvert^p)^{\frac{1}{p}})$	$\lvert\ln a\rvert^p$	$p > 0$	$i_{\min}(a,b)$	ab when $p = 1$	$\min(a,b)$
Schweizer & Sklar 4	$\frac{ab}{[a^p+b^p-a^pb^p]^{\frac{1}{p}}}$	$a^{-p}-1$	$p > 0$	ab	$\frac{ab}{a+b-ab}$, when $p = 1$	$\min(a,b)$
Yager [1980f]	$1-\min\left\{1, [(1-a)^w+(1-b)^w]^{\frac{1}{w}}\right\}$	$(1-a)^w$	$w > 0$	$i_{\min}(a,b)$	$\max(0, a+b-1)$ when $w = 1$	$\min(a,b)$
Dubois & Prade [1980]	$\frac{ab}{\max(a,b,\alpha)}$		$\alpha \in [0,1]$	$\min(a,b)$	ab when $\alpha = 1$	
Weber [1983]	$\max\left(0, \frac{a+b+\lambda ab-1}{1+\lambda}\right)$	$\frac{1}{\lambda}\ln[1+\lambda(1-a)]$	$\lambda > -1$	$\max(0, a+b-1)$	$i_{\min}(a,b)$ as $\lambda \to -1$; $\max[0, (a+b+ab-1)/2]$ when $\lambda = 1$.	ab
Yu [1985]	$\max[0, (1+\lambda)(a+b-1)-\lambda ab]$	$\frac{1}{\lambda}\ln\frac{1+\lambda}{1+\lambda a}$	$\lambda > -1$	$\max(0, a+b-1)$	ab as $\lambda \to -1$; $\max[0, 2(a+b-ab/2-1)]$ when $\lambda = 1$.	$i_{\min}(a,b)$

TABLE 3.3 SOME CLASSES OF FUZZY UNIONS (t-CONORMS)

Reference	Formula $u(a,b)$	Increasing generator $g(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{-\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a}-1\right)^{-\lambda}$	$\lambda > 0$	$u_{\max}(a,b)$	$\frac{a+b-2ab}{1-ab}$ when $\lambda=1$	$\max(a,b)$
Frank [1979]	$1-\log_s\left[1+\frac{(s^{1-a}-1)(s^{1-b}-1)}{s-1}\right]$	$-\ln\left(\frac{s^{1-a}-1}{s-1}\right)$	$s>0, s\neq 1$	$\max(a,b)$	$a+b-ab$ as $s\to 1$	$\min(1,a+b)$
Hamacher [1978]	$\frac{a+b+(r-2)ab}{r+(r-1)ab}$	$-\ln\left(\frac{1-a}{r+(1-r)(1-a)}\right)$	$r>0$	$\frac{a+b-2ab}{1-ab}$	$a+b-ab$ when $r=1$	$u_{\max}(a,b)$
Schweizer & Sklar 1 [1963]	$1-\{\max(0,(1-a)^p+(1-b)^p-1)\}^{\frac{1}{p}}$	$1-(1-a)^p$	$p\neq 0$	$a+b-ab$	$\min(1,a+b)$, when $p=1$; $\frac{a+b-2ab}{1-ab}$, when $p=-1$.	$u_{\max}(a,b)$ as $p\to\infty$; $\min(a,b)$ as $p\to-\infty$.
Schweizer & Sklar 2	$[a^p+b^p-a^pb^p]^{\frac{1}{p}}$	$\ln[1-a^p]^{\frac{1}{p}}$	$p>0$	$u_{\max}(a,b)$	$a+b-ab$ when $p=1$	$\max(a,b)$
Schweizer & Sklar 3	$1-\exp(-(\lvert\ln(1-a)\rvert^p+\lvert\ln(1-b)\rvert^p)^{\frac{1}{p}})$	$\lvert\ln(1-a)\rvert^p$	$p>0$	$u_{\max}(a,b)$	$a+b-ab$ when $p=1$	$\max(a,b)$
Schweizer & Sklar 4	$1-\frac{(1-a)(1-b)}{[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}}$	$(1-a)^{-p}-1$	$p>0$	$a+b-ab$	$\min\left(1,\frac{a+b}{1-ab}\right)$ when $p=1$	$\max(a,b)$
Yager [1980f]	$\min\left[1,(a^w+b^w)^{\frac{1}{w}}\right]$	a^w	$w>0$	$u_{\max}(a,b)$	$\min(1,a+b)$ when $w=1$	$\max(a,b)$
Dubois & Prade [1980]	$1-\frac{(1-a)(1-b)}{\max((1-a),(1-b),\alpha)}$		$\alpha\in[0,1]$	$\max(a,b)$	$a+b-ab$ when $\alpha=1$	
Weber [1983]	$\min\left(1,a+b-\frac{\lambda}{1-\lambda}ab\right)$	$\frac{1}{\lambda}\ln\frac{1+\lambda}{1+\lambda(1-a)}$	$\lambda>-1$	$\min(1,a+b)$	$u_{\max}(a,b)$ as $\lambda\to-1$; $\min(1,(a+b-ab/2)$ when $\lambda=1$.	$a+b-ab$
Yu [1985]	$\min(1,a+b+\lambda ab)$	$\frac{1}{\lambda}\ln(1+\lambda a)$	$\lambda>-1$	$\min(1,a+b)$	$a+b-ab$ as $\lambda\to-1$; $\min(1,a+b+ab)$ when $\lambda=1$.	$u_{\max}(a,b)$