Fuzzy Sets, Uncertainty, and Information

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PREFACE

It has increasingly been recognized that our society is undergoing a significant transformation, usually described as a transition from an industrial to an information society. There is little doubt that this transition is strongly connected with the emergence and development of computer technology and with the associated intellectual activities resulting in new fields of inquiry such as systems science, information science, decision analysis, or artificial intelligence.

Advances in computer technology have been steadily extending our capabilities for coping with systems of an increasingly broad range, including systems that were previously intractable to us by virtue of their nature and complexity. While the level of complexity we can manage continues to increase, we begin to realize that there are fundamental limits in this respect. As a consequence, we begin to understand that the necessity for simplification of systems, many of which have become essential for characterizing certain currently relevant problem situations, is often unavoidable. In general, a good simplification should minimize the loss of information relevant to the problem of concern. Information and complexity are thus closely interrelated.

One way of simplifying a very complex system—perhaps the most significant one—is to allow some degree of uncertainty in its description. This entails an appropriate aggregation or summary of the various entities within the system. Statements obtained from this simplified system are less precise (certain), but their relevance to the original system is fully maintained. That is, the information loss that is necessary for reducing the complexity of the system to a manageable level is expressed in uncertainty. The concept of uncertainty is thus connected with both complexity and information.

It is now realized that there are several fundamentally different types of uncertainty and that each of them plays a distinct role in the simplification problem. A mathematical formulation within which these various types of uncertainty can be properly characterized and investigated is now available in terms of the theory of fuzzy sets and fuzzy measures.

The primary purpose of this book is to bring this new mathematical formalism into the education system, not merely for its own sake, but as a basic framework for characterizing the full scope of the concept of uncertainty and its relationship to the increasingly important concepts of information and complexity. It should be stressed that these concepts arise in virtually all fields of inquiry; the usefulness of the mathematical framework presented in this book thus transcends the artificial boundaries of the various areas and specializations in the sciences and professions. This book is intended, therefore, to make an understanding of this mathematical formalism accessible to students and professionals in a broad range of disciplines. It is written specifically as a text for a one-semester course at the graduate or upper division undergraduate level that covers the various issues of uncertainty, information, and complexity from a broad perspective based on the formalism of fuzzy set theory. It is our hope that this book will encourage the initiation of new courses of this type in the various programs of higher education as well as in programs of industrial and continuing education. The book is, in fact, a by-product of one such graduate level course, which has been taught at the State University of New York at Binghamton for the last three years.

No previous knowledge of fuzzy set theory or information theory is required for an understanding of the material in this book, thus making it a virtually self-contained text. Although we assume that the reader is familiar with the basic notions of classical (nonfuzzy) set theory, classical (two-valued) logic, and probability theory, the fundamentals of these subject areas are briefly overviewed in the book. In addition, the basic ideas of classical information theory (based on the Hartley and Shannon information measures) are also introduced. For the convenience of the reader, we have included in Appendix B a glossary of the symbols most frequently used in the text.

Chapters 1–3 cover the fundamentals of fuzzy set theory and its connection with fuzzy logic. Particular emphasis is given to a comprehensive coverage of operations on fuzzy sets (Chap. 2) and to various aspects of fuzzy relations (Chap. 3). The concept of general fuzzy measures is introduced in Chap. 4, but the main focus of this chapter is on the dual classes of belief and plausibility measures along with some of their special subclasses (possibility, necessity, and probability measures); this chapter does not require a previous reading of Chapters 2 and 3. Chapter 5 introduces the various types of uncertainty and discusses their relation to information and complexity. Measures of the individual types of uncertainty are investigated in detail and proofs of the uniqueness of some of these are included in Appendix A. The classical information theory (based on the Hartley and Shannon measures of uncertainty) is overviewed, but the major emphasis is given to the new measures of uncertainty and information that have emerged from fuzzy set theory. While Chapters 1–5 focus on theoretical developments, Chap. 6 offers a brief look at some of the areas in which successful applications of this mathematical formalism have been made. Each section of Chap. 6 gives a brief overview of a major area of application along with some specific illustrative examples. We

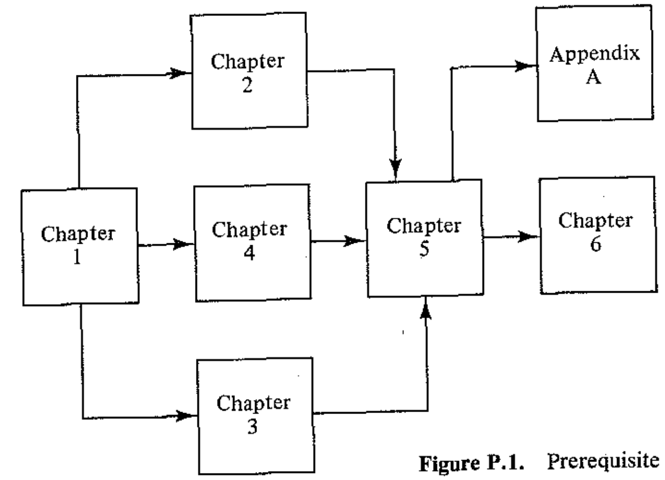

Figure P.1. Prerequisite dependencies among chapters of this book.

have attempted to provide the reader with a flavor of the numerous and diverse areas of application of fuzzy set theory and information theory without attempting an exhaustive study of each one. Ample references are included, however, which will allow the interested reader to pursue further study in the application area of concern.

The prerequisite dependencies among the individual chapters are expressed by the diagram in Fig. P.1. It is clear that the reader has some flexibility in studying the material; for instance, the chapters may be read in order, or the study of Chap. 4 may preceed that of Chaps. 2 and 3.

In order to avoid interruptions in the main text, virtually all bibliographical, historical, and other side remarks are incorporated into the notes that follow each individual chapter. These notes are uniquely numbered and are only occasionally referred to in the text.

When the book is used at the undergraduate level, coverage of some or all of the proofs of the various mathematical theorems may be omitted, depending on the background of the students. At the graduate level, on the other hand, we encourage coverage of most of these proofs in order to effect a deeper understanding of the material. In all cases, the relevance of the material to the specific area of student interest or study can be emphasized with additional application-oriented readings; the notes to Chap. 6 contain annotated references to guide in the selection of such readings from the literature.

Chapters 1–5 are each followed by a set of exercises, which are intended to enhance an understanding of the material presented in the chapter. The solutions to a selected subset of these exercises are provided in the instructor's manual; the remaining exercises are left unanswered so as to be suitable for examination use. Further suggestions for the use of this book in the teaching context can be found in the instructor's manual.

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1

Crisp Sets and Fuzzy Sets

1.1 INTRODUCTION

The process and progress of knowledge unfolds into two stages: an attempt to know the character of the world and a subsequent attempt to know the character of knowledge itself. The second reflective stage arises from the failures of the first; it generates an inquiry into the possibility of knowledge and into the limits of that possibility. It is in this second stage of inquiry that we find ourselves today. As a result, our concerns with knowledge, perceptions of problems and attempts at solutions are of a different order than in the past. We want to know not only specific facts or truths but what we can and cannot know, what we do and do not know, and how we know at all. Our problems have shifted from questions of how to cope with the world (how to provide ourselves with food, shelter, and so on), to questions of how to cope with knowledge (and ignorance) itself. Ours has been called an "information society," and a major portion of our economy is devoted to the handling, processing, selecting, storing, disseminating, protecting, collecting, analyzing, and sorting of information, our best tool for this being, of course, the computer.

Our problems are seen in terms of decision, management, and prediction; solutions are seen in terms of faster access to more information and of increased aid in analyzing, understanding and utilizing the information that is available and in coping with the information that is not. These two elements, large amounts of information coupled with large amounts of uncertainty, taken together constitute the ground of many of our problems today: complexity. As we become aware of how much we know and of how much we do not know, as information and uncertainty themselves become the focus of our concern, we begin to see our prob-

The fact that complexity itself includes both the element of how much we know, or how well we can describe, and the element of how much we do not know, or how uncertain we are, can be illustrated with the simple example of driving a car. We can probably agree that driving a car is (at least relatively) complex. Further, driving a standard transmission or stick-shift car is more complex than driving a car with an automatic transmission, one index of this being that more description is needed to cover adequately our knowledge of driving in the former case than in the latter. Thus, because more knowledge is involved in the driving of a standard-transmission car (we must know, for instance, the revolutions per minute of the engine and how to use the clutch), it is more complex. However, the complexity of driving also involves the degree of our uncertainty; for example, we do not know precisely when we will have to stop or swerve to avoid an obstacle. As our uncertainty increases—for instance, in heavy traffic or on unfamiliar roads—so does the complexity of the task. Thus, our perception of complexity increases both when we realize how much we know and when we realize how much we do not know.

How do we manage to cope with complexity as well as we do, and how could we manage to cope better? The answer seems to lie in the notion of simplifying complexity by making a satisfactory trade-off or compromise between the information available to us and the amount of uncertainty we allow. One option is to increase the amount of allowable uncertainty by sacrificing some of the precise information in favor of a vague but more robust summary. For instance, instead of describing the weather today in terms of the exact percentage of cloud cover (which would be much too complex), we could just say that it is sunny, which is more uncertain and less precise but more useful. In fact, it is important to realize that the imprecision or vagueness that is characteristic of natural language does not necessarily imply a loss of accuracy or meaningfulness. It is, for instance, generally more meaningful to give travel directions in terms of city blocks than in terms of inches, although the former is much less precise than the latter. It is also more accurate to say that it is usually warm in the summer than to say that it is usually 72° in the summer. In order for a term such as *sunny* to accomplish the desired introduction of vagueness, however, we cannot use it to mean precisely 0 percent cloud cover. Its meaning is not totally arbitrary, however; a cloud cover of 100 percent is not sunny and neither, in fact, is a cloud cover of 80 percent. We can accept certain intermediate states, such as 10 or 20 percent cloud cover, as sunny. But where do we draw the line? If, for instance, any cloud cover of 25 percent or less is considered sunny, does this mean that a cloud cover of 26 percent is not? This is clearly unacceptable since 1 percent of cloud cover hardly seems like a distinguishing characteristic between sunny and not sunny. We could, therefore, add a qualification that any amount of cloud cover 1 percent greater than a cloud cover already considered to be sunny (that is, 25 percent or less) will also be labeled as sunny. We can see, however, that this definition eventually leads us to accept all degrees of cloud cover as sunny, no matter how gloomy the weather looks! In order to resolve this paradox, the term *sunny* may introduce vagueness by allowing some sort of gradual transition from degrees of cloud cover that are considered to be sunny and those that are

not. This is, in fact, precisely the basic concept of the *fuzzy set*, a concept that is both simple and intuitively pleasing and that forms, in essence, a generalization of the classical or *crisp set*.

The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and nonmembers (those that certainly do not). A sharp, unambiguous distinction exists between the members and nonmembers of the class or category represented by the crisp set. Many of the collections and categories we commonly employ, however (for instance, in natural language), such as the classes of tall people, expensive cars, highly contagious diseases, numbers much greater than 1, or sunny days, do not exhibit this characteristic. Instead, their boundaries seem vague, and the transition from member to nonmember appears gradual rather than abrupt. Thus, the fuzzy set introduces vagueness (with the aim of reducing complexity) by eliminating the sharp boundary dividing members of the class from nonmembers. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set. Thus, individuals may belong in the fuzzy set to a greater or lesser degree as indicated by a larger or smaller membership grade. These membership grades are very often represented by real-number values ranging in the closed interval between 0 and 1. Thus, a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0 percent, .8 to a cloud cover of 20 percent, .4 to a cloud cover of 30 percent and 0 to a cloud cover of 75 percent. These grades signify the degree to which each percentage of cloud cover approximates our subjective concept of *sunny*, and the set itself models the semantic flexibility inherent in such a common linguistic term. Because full membership and full nonmembership in the fuzzy set can still be indicated by the values of 1 and 0, respectively, we can consider the crisp set to be a restricted case of the more general fuzzy set for which only these two grades of membership are allowed.

Research on the theory of fuzzy sets has been abundant, and in this book we present an introduction to the major developments of the theory. There are, however, several types of uncertainty other than the type represented by the fuzzy set. The classical probability theory, in fact, represents one of these alternative and distinct forms of uncertainty. Understanding these various types of uncertainty and their relationships with information and complexity is currently an area of active and promising research. Therefore, in addition to offering a thorough introduction to the fuzzy set theory, this book provides an overview of the larger framework of issues of uncertainty, information, and complexity and places the fuzzy set theory within this framework of mathematical explorations.

In addition to presenting the theoretical foundations of fuzzy set theory and associated measures of uncertainty and information, the last chapter of this book offers a glimpse at some of the successful applications of this new conceptual framework to real-world problems. As general tools for dealing with complexity independent of the particular content of concern, the theory of fuzzy sets and the various mathematical representations and measurements of uncertainty and information have a virtually unrestricted applicability. Indeed, possibilities for application include any field that examines how we process or act on information, make decisions, recognize patterns, or diagnose problems or any field in which the complexity of the necessary knowledge requires some form of simplification. Successful applications have, in fact, been made in fields as numerous and diverse as engineering, psychology, artificial intelligence, medicine, ecology, decision theory, pattern recognition, information retrieval, sociology, and meteorology. Few fields remain, in fact, in which conceptions of the major problems and obstacles have not been reformulated in terms of the handling of information and uncertainty. While the diversity of successful applications has thus been expanding rapidly, the theory of fuzzy sets in particular and the mathematics of uncertainty and information in general have been achieving a secure identity as valid and useful extensions of classical mathematics. They will undoubtedly continue to constitute an important framework for further investigations into rigorous representations of uncertainty, information, and complexity.

1.2 CRISP SETS: AN OVERVIEW

This text is devoted to an examination of fuzzy sets as a broad conceptual framework for dealing with uncertainty and information. The reader's familiarity with the basic theory of crisp sets is assumed. Therefore, this section is intended to serve simply to refresh the basic concepts of crisp sets and to introduce notation and terminology useful for our discussion of fuzzy sets.

Throughout this book, sets are denoted by capital letters and their members by lower-case letters. The letter X denotes the universe of discourse, or *universal set*. This set contains all the possible elements of concern in each particular context or application from which sets can be formed. Unless otherwise stated, X is assumed in this text to contain a finite number of elements.

To indicate that an individual object x is a *member* or *element* of a set A, we write

$$x \in A.$$

Whenever x is not an element of a set A, we write

$$x \notin A.$$

A set can be described either by naming all its members (the *list method*) or by specifying some well-defined properties satisfied by the members of the set (the *rule method*). The list method, however, can be used only for finite sets. The set A whose members are $a_1, a_2, \ldots, a_n$ is usually written as

$$A = \{a_1, a_2, \ldots, a_n\},$$

and the set B whose members satisfy the properties $P_1, P_2, \ldots, P_n$ is usually

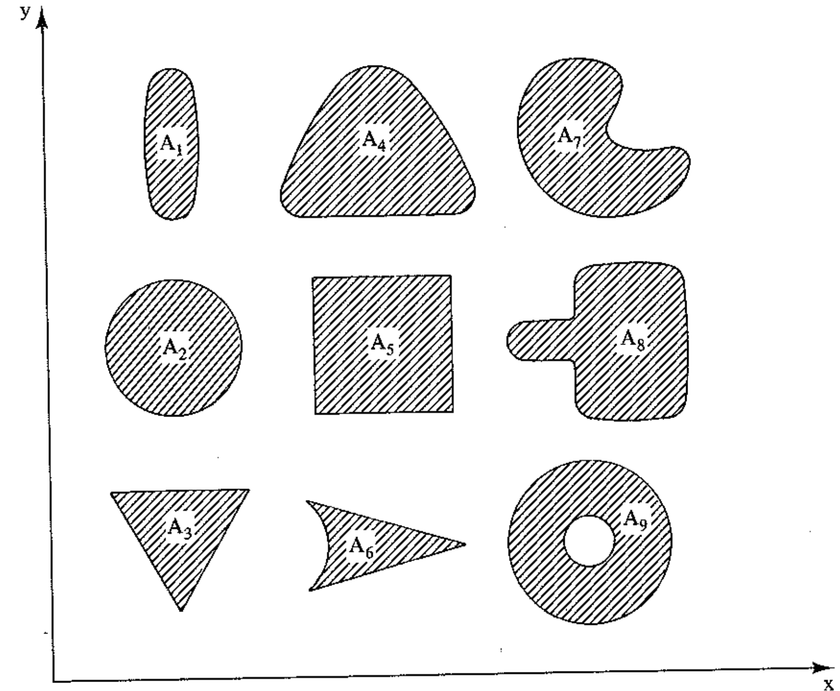


Figure 1.1. Example of sets in $\mathbb{R}^2$ that are either convex (A_1–A_5) or nonconvex (A_6–A_9).

written as

$$B = \{b \mid b \text{ has properties } P_1, P_2, \ldots, P_n\},$$

where the symbol | denotes the phrase "such that."

An important and frequently used universal set is the set of all points in the n-dimensional Euclidean vector space $\mathbb{R}^n$ (i.e., all n-tuples of real numbers). Sets defined in terms of $\mathbb{R}^n$ are often required to possess a property referred to as convexity. A set A in $\mathbb{R}^n$ is called *convex* if, for every pair of points*

$$\mathbf{r} = (r_i \mid i \in \mathbb{N}_n) \quad \text{and} \quad \mathbf{s} = (s_i \mid i \in \mathbb{N}_n)$$

in A and every real number λ between 0 and 1, exclusively, the point

$$\mathbf{t} = (\lambda r_i + (1 - \lambda)s_i \mid i \in \mathbb{N}_n)$$

is also in A. In other words, a set A in $\mathbb{R}^n$ is convex if, for every pair of points $\mathbf{r}$ and $\mathbf{s}$ in A, all points located on the straight line segment connecting $\mathbf{r}$ and $\mathbf{s}$ are also in A. Examples of convex and nonconvex sets in $\mathbb{R}^2$ are given in Fig. 1.1.

* $\mathbb{N}$ subscripted by a positive integer is used in this text to denote the set of all integers from 1 through the value of the subscript; that is, $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

A set whose elements are themselves sets is often referred to as a *family of sets*. It can be defined in the form

$$\{A_i \mid i \in I\},$$

where i and I are called the *set identifier* and the *identification set*, respectively. Because the index i is used to reference the sets A_i, the family of sets is also called an *indexed set*.

If every member of set A is also a member of set B—that is, if $x \in A$ implies $x \in B$—then A is called a *subset* of B, and this is written as

$$A \subseteq B.$$

Every set is a subset of itself and every set is a subset of the universal set. If $A \subseteq B$ and $B \subseteq A$, then A and B contain the same members. They are then called *equal* sets; this is denoted by

$$A = B.$$

To indicate that A and B are not equal, we write

$$A \neq B.$$

If both $A \subseteq B$ and $A \neq B$, then B contains at least one individual that is not a member of A. In this case, A is called a *proper subset* of B, which is denoted by

$$A \subset B.$$

The set that contains no members is called the *empty set* and is denoted by $\varnothing$. The empty set is a subset of every set and is a proper subset of every set except itself.

The process by which individuals from the universal set X are determined to be either members or nonmembers of a set can be defined by a *characteristic*, or *discrimination, function*. For a given set A, this function assigns a value $\mu_A(x)$ to every $x \in X$ such that

$$\mu_A(x) = \begin{cases} 1 & \text{if and only if } x \in A, \\ 0 & \text{if and only if } x \notin A. \end{cases}$$

Thus, the function maps elements of the universal set to the set containing 0 and 1. This can be indicated by

$$\mu_A : X \to \{0, 1\}.$$

The number of elements that belong to a set A is called the *cardinality* of the set and is denoted by $|A|$. A set that is defined by the rule method may contain an infinite number of elements.

The family of sets consisting of all the subsets of a particular set A is referred to as the *power set* of A and is indicated by $\mathcal{P}(A)$. It is always the case that

$$|\mathcal{P}(A)| = 2^{|A|}.$$

The *relative complement* of a set A with respect to set B is the set containing

all the members of B that are not also members of A. This can be written $B - A$. Thus,

$$B - A = \{x \mid x \in B \quad \text{and} \quad x \notin A\}.$$

If the set B is the universal set, the complement is absolute and is usually denoted by $\bar{A}$. Complementation is always *involutive*; that is, taking the complement of a complement yields the original set, or

$$\bar{\bar{A}} = A.$$

The absolute complement of the empty set equals the universal set, and the absolute complement of the universal set equals the empty set. That is,

$$\bar{\varnothing} = X,$$

and

$$\bar{X} = \varnothing.$$

The *union* of sets A and B is the set containing all the elements that belong either to set A alone, to set B alone, or to both set A and set B. This is denoted by $A \cup B$. Thus,

$$A \cup B = \{x \mid x \in A \quad \text{or} \quad x \in B\}.$$

The union operation can be generalized for any number of sets. For a family of sets $\{A_i \mid i \in I\}$, this is defined as

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}.$$

The union of any set with the universal set yields the universal set, whereas the union of any set with the empty set yields the set itself. We can write this as

$$A \cup X = X$$

and

$$A \cup \varnothing = A.$$

Because all the elements of the universal set necessarily belong either to a set A or to its absolute complement, $\bar{A}$, the union of A and $\bar{A}$ yields the universal set. Thus,

$$A \cup \bar{A} = X.$$

This property is usually called the *law of excluded middle*.

The *intersection* of sets A and B is the set containing all the elements belonging to both set A and set B. It is denoted by $A \cap B$. Thus,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The generalization of the intersection for a family of sets $\{A_i \mid i \in I\}$ is defined as

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$

The intersection of any set with the universal set yields the set itself, and the intersection of any set with the empty set yields the empty set. This can be indicated by writing

$$A \cap X = A$$

and

$$A \cap \varnothing = \varnothing.$$

Since a set and its absolute complement by definition share no elements, their intersection yields the empty set. Thus,

$$A \cap \bar{A} = \varnothing.$$

This property is usually called the *law of contradiction.*

Any two sets A and B are *disjoint* if they have no elements in common, that is, if

$$A \cap B = \varnothing.$$

It follows directly from the law of contradiction that a set and its absolute complement are always disjoint.

A collection of pairwise disjoint nonempty subsets of a set A is called a *partition* on A if the union of these subsets yields the original set A. We denote a partition on A by the symbol $\pi(A)$. Formally,

$$\pi(A) = \{A_i \mid i \in I, \quad A_i \subseteq A\},$$

where $A_i \neq \varnothing$, is a partition on A if and only if

$$A_i \cap A_j = \varnothing.$$

for each pair $i \neq j$, $i, j \in I$, and

$$\bigcup_{i \in I} A_i = A.$$

Thus, each element of A belongs to one and only one of the subsets forming the partition.

There are several important properties that are satisfied by the operations of union, intersection and complement. Both union and intersection are *commutative,* that is, the result they yield is not affected by the order of their operands. Thus,

$$A \cup B = B \cup A,$$

$$A \cap B = B \cap A.$$

Union and intersection can also be applied pairwise in any order without altering the result. We call this property *associativity* and express it by the equations

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C),$$

$$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C),$$

where the operations in parentheses are performed first.

Because the union and intersection of any set with itself yields that same set, we say that these two operations are *idempotent.* Thus,

$$A \cup A = A,$$

$$A \cap A = A.$$

The *distributive law* is also satisfied by union and intersection in the following ways:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Finally, *DeMorgan's law* for union, intersection, and complement states that the complement of the intersection of any two sets equals the union of their complements. Likewise, the complement of the union of two sets equals the intersection of their complements. This can be written as

$$\overline{A \cap B} = \bar{A} \cup \bar{B},$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}.$$

These and some additional properties are summarized in Table 1.1. Note that all the equations in this table that involve the set union and intersection are arranged in pairs. The second equation in each pair can be obtained from the first by replacing $\varnothing$, $\cup$, and $\cap$ with X, $\cap$, and $\cup$, respectively, and vice versa. We

TABLE 1.1. PROPERTIES OF CRISP SET OPERATIONS.

Involution	$\bar{\bar{A}} = A$
Commutativity	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$ $A \cap A = A$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Absorption of complement	$A \cup (\bar{A} \cap B) = A \cup B$ $A \cap (\bar{A} \cup B) = A \cap B$
Absorption by X and $\varnothing$	$A \cup X = X$ $A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$ $A \cap X = A$
Law of contradiction	$A \cap \bar{A} = \varnothing$
Law of excluded middle	$A \cup \bar{A} = X$
DeMorgan's laws	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$

are thus concerned with pairs of dual equations. They exemplify a *general principle of duality*: for each valid equation in set theory that is based on the union and intersection operations, there corresponds a dual equation, also valid, that is obtained by the above specified replacement.

1.3 THE NOTION OF FUZZY SETS

As defined in the previous section, the characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote higher degrees of set membership. Such a function is called a *membership function* and the set defined by it a *fuzzy set*.

Let X denote a universal set. Then, the membership function μ_A by which a fuzzy set A is usually defined has the form

$$\mu_A : X \rightarrow [0, 1],$$

where [0, 1] denotes the interval of real numbers from 0 to 1, inclusive.

For example, we can define a possible membership function for the fuzzy set of real numbers close to 0 as follows:

$$\mu_A(x) = \frac{1}{1 + 10x^2}.$$

The graph of this function is pictured in Fig. 1.2. Using this function, we can determine the membership grade of each real number in this fuzzy set, which

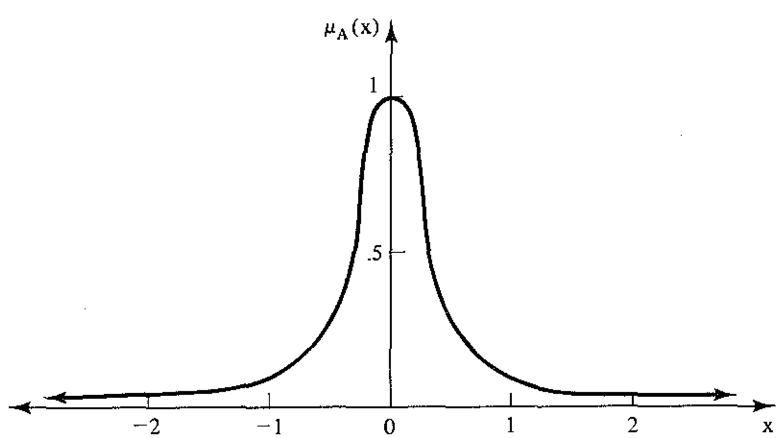


Figure 1.2. A possible membership function of the fuzzy set of real numbers close to zero.

signifies the degree to which that number is close to 0. For instance, the number 3 is assigned a grade of .01, the number 1 a grade of .09, the number .25 a grade of .62, and the number 0 a grade of 1. We might intuitively expect that by performing some operation on the function corresponding to the set of numbers close to 0, we could obtain a function representing the set of numbers very close to 0. One possible way of accomplishing this is to square the function, that is,

$$\mu_A(x) = \left(\frac{1}{1 + 10x^2}\right)^2.$$

We could also generalize this function to a family of functions representing the set of real numbers close to any given number a as follows:

$$\mu_A(x) = \frac{1}{1 + 10(x - a)^2}.$$

Although the range of values between 0 and 1, inclusive, is the one most commonly used for representing membership grades, any arbitrary set with some natural full or partial ordering can in fact be used. Elements of this set are not required to be numbers as long as the ordering among them can be interpreted as representing various strengths of membership degree. This generalized membership function has the form

$$\mu_A : X \rightarrow L,$$

where L denotes any set that is at least partially ordered. Since L is most frequently a lattice, fuzzy sets defined by this generalized membership grade function are called *L-fuzzy sets*, where L is intended as an abbreviation for *lattice*. (The full definitions of partial ordering, total ordering, and lattice are given in Sec. 3.6.) L-fuzzy sets are important in certain applications, perhaps the most important being those in which $L = [0, 1]^n$. The symbol $[0, 1]^n$ is a shorthand notation of the Cartesian product

$$\underbrace{[0, 1] \times [0, 1] \times \cdots \times [0, 1]}_{n \text{ times}}$$

(see Sec. 3.1). Although the set [0, 1] is totally ordered, sets $[0, 1]^n$ for any $n \geq 2$ are ordered only partially. For example, any two pairs $(a_1, b_1) \in [0, 1]^2$ and $(a_2, b_2) \in [0, 1]^2$ are not comparable (ordered) whenever $a_1 < a_2$ and $b_1 > b_2$.

A few examples in this book demonstrate the utility of L-fuzzy sets. For the most part, however, our discussions and examples focus on the classical representation of membership grades using real-number values in the interval [0, 1].

Fuzzy sets are often incorrectly assumed to indicate some form of probability. Despite the fact that they can take on similar values, it is important to realize that membership grades are *not* probabilities. One immediately apparent difference is that the summation of probabilities on a finite universal set must equal 1, while there is no such requirement for membership grades. A more thorough discussion of the distinction between these two expressions of uncertainty is made in Chap. 4.

A further distinction must be drawn between the concept of a fuzzy set and another representation of uncertainty known as the *fuzzy measure*. Given a particular element of a universal set of concern whose membership in the various *crisp subsets* of this universal set is not known with certainty, a fuzzy measure g assigns a graded value to each of these crisp subsets, which indicates the degree of evidence or subjective certainty that the element belongs in the subset. Thus, the fuzzy measure is defined by the function

$$g: \mathcal{P}(X) \rightarrow [0, 1],$$

which satisfies certain properties. Fuzzy measures are covered in Chap. 4.

The difference between fuzzy sets and fuzzy measures can be briefly illustrated by an example. For any particular person under consideration, the evidence of age that would be necessary to place that person with certainty into the group of people in their twenties, thirties, forties, or fifties may be lacking. Note that these sets are crisp; there is no fuzziness associated with their boundaries. The set assigned the highest value in this particular fuzzy measure is our best guess of the person's age; the next highest value indicates the degree of certainty associated with our next best guess, and so on. Better evidence would result in a higher value for the best guess until absolute proof would allow us to assign a grade of 1 to a single crisp set and 0 to all the others. This can be contrasted with a problem formulated in terms of fuzzy sets in which we know the person's age but must determine to what degree he or she is considered, for instance, "old" or "young." Thus, the type of uncertainty represented by the fuzzy measure should not be confused with that represented by fuzzy sets. Chapter 4 contains a further elaboration of this distinction.

Obviously, the usefulness of a fuzzy set for modeling a conceptual class or a linguistic label depends on the appropriateness of its membership function. Therefore, the practical determination of an accurate and justifiable function for any particular situation is of major concern. The methods proposed for accomplishing this have been largely empirical and usually involve the design of experiments on a test population to measure subjective perceptions of membership degrees for some particular conceptual class. There are various means for implementing such measurements. Subjects may assign actual membership grades, the statistical response pattern for the true or false question of set membership may be sampled, or the time of response to this question may be measured, where shorter response times are taken to indicate higher subjective degrees of membership. Once these data are collected, there are several ways in which a membership function reflecting the results can be derived. Since many applications for fuzzy sets involve modeling the perceptions of a limited population for specified concepts, these methods of devising membership functions are, on the whole, quite useful. More detailed examples of some applied derivation methods are discussed in Chap. 6.

The accuracy of any membership function is necessarily limited. In addition, it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers. Although this does not pose a serious problem for many applications, it is never-

theless possible to extend the concept of the fuzzy set to allow the distinction between grades of membership to become blurred. Sets described in this way are known as *type 2 fuzzy sets*. By definition, a type 1 fuzzy set is an ordinary fuzzy set and the elements of a type 2 fuzzy set have membership grades that are themselves type 1 (i.e., ordinary) fuzzy sets defined on some universal set Y. For example, if we define a type 2 fuzzy set "*intelligent*," membership grades assigned to elements of X (a population of human beings) might be type 1 fuzzy sets such as *average, below average, superior, genius,* and so on. Note that every fuzzy set of type 2 is an L-fuzzy set. When the membership grades employed in the definition of a type 2 fuzzy set are themselves type 2 fuzzy sets, the set is viewed as a type 3 fuzzy set. In the same way, higher types of fuzzy sets are defined.

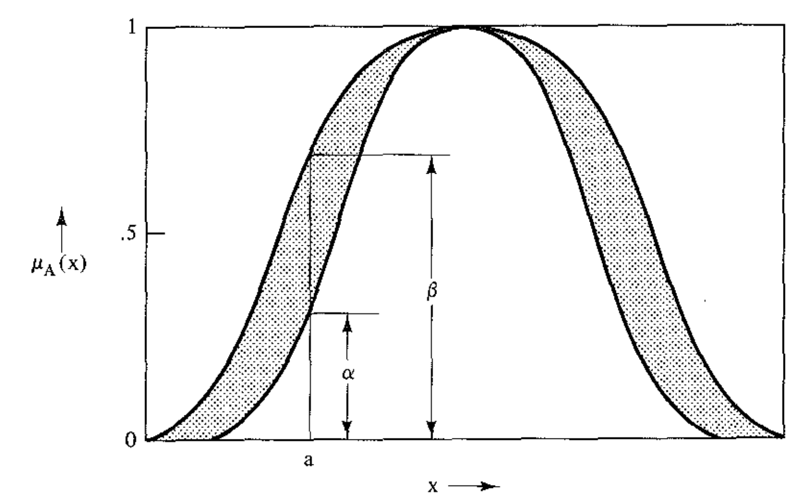
A different extension of the fuzzy set concept involves creating fuzzy subsets of a universal set whose elements are fuzzy sets. These fuzzy sets are known as *level k fuzzy sets*, where k indicates the depth of nesting. For instance, the elements of a level 3 fuzzy set are level 2 fuzzy sets whose elements are in turn level 1 fuzzy sets. One example of a level 2 fuzzy set is the collection of desired attributes for a new car, where elements from the universe of discourse are ordinary (level 1) fuzzy sets such as *inexpensive, reliable, sporty,* and so on.

Given a crisp universal set X, let $\tilde{\mathcal{P}}(X)$ denote the set of all fuzzy subsets of X and let $\tilde{\mathcal{P}}^k(X)$ be defined recursively by the equation

$$\tilde{\mathcal{P}}^k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}^{k-1}(X)),$$

for all integers $k \geq 2$. Then, fuzzy sets of level k are formally defined by membership functions of the form

$$\mu_A: \tilde{\mathcal{P}}^{k-1}(X) \rightarrow [0, 1],$$



or, when extended to L-fuzzy sets, by functions

$$\mu_A : \tilde{\mathcal{P}}^{k-1}(X) \rightarrow L.$$

The requirement for a precise membership function can also be relaxed by allowing values $\mu_A(x)$ to be intervals of real numbers in [0, 1] rather than single numbers. Fuzzy sets of this sort are called *interval-valued fuzzy sets*. They are formally defined by membership functions of the form

$$\mu_A : X \rightarrow \mathcal{P}([0, 1]).$$

where $\mu_A(x)$ is a closed interval in [0, 1] for each $x \in X$. An example of this kind of membership function is given in Fig. 1.3; for each x, $\mu_A(x)$ is represented by the segment between the two curves. It is clear that the concept of interval-valued fuzzy sets can be extended to L-fuzzy sets by replacing [0, 1] with a partially ordered set L and requiring that, for each $x \in X$, $\mu_A(x)$ be a segment of totally ordered elements in L.

1.4 BASIC CONCEPTS OF FUZZY SETS

This section introduces some of the basic concepts and terminology of fuzzy sets. Many of these are extensions and generalizations of the basic concepts of crisp sets, but others are unique to the fuzzy set framework. To illustrate some of the concepts, we consider the membership grades of the elements of a small universal set in four different fuzzy sets as listed in Table 1.2 and graphically expressed in Fig. 1.4. Here the crisp universal set X of ages that we have selected is

$$X = \{5, 10, 20, 30, 40, 50, 60, 70, 80\},$$

and the fuzzy sets labeled as *infant, adult, young,* and *old* are four of the elements of the power set containing all possible fuzzy subsets of X, which is denoted by $\tilde{\mathcal{P}}(X)$.

The *support* of a fuzzy set A in the universal set X is the crisp set that contains all the elements of X that have a nonzero membership grade in A. That

TABLE 1.2. EXAMPLES OF FUZZY SETS.

Elements (ages)	Infant	Adult	Young	Old
5	0	0	1	0
10	0	0	1	0
20	0	.8	.8	.1
30	0	1	.5	.2
40	0	1	.2	.4
50	0	1	.1	.6
60	0	1	0	.8
70	0	1	0	1
80	0	1	0	1

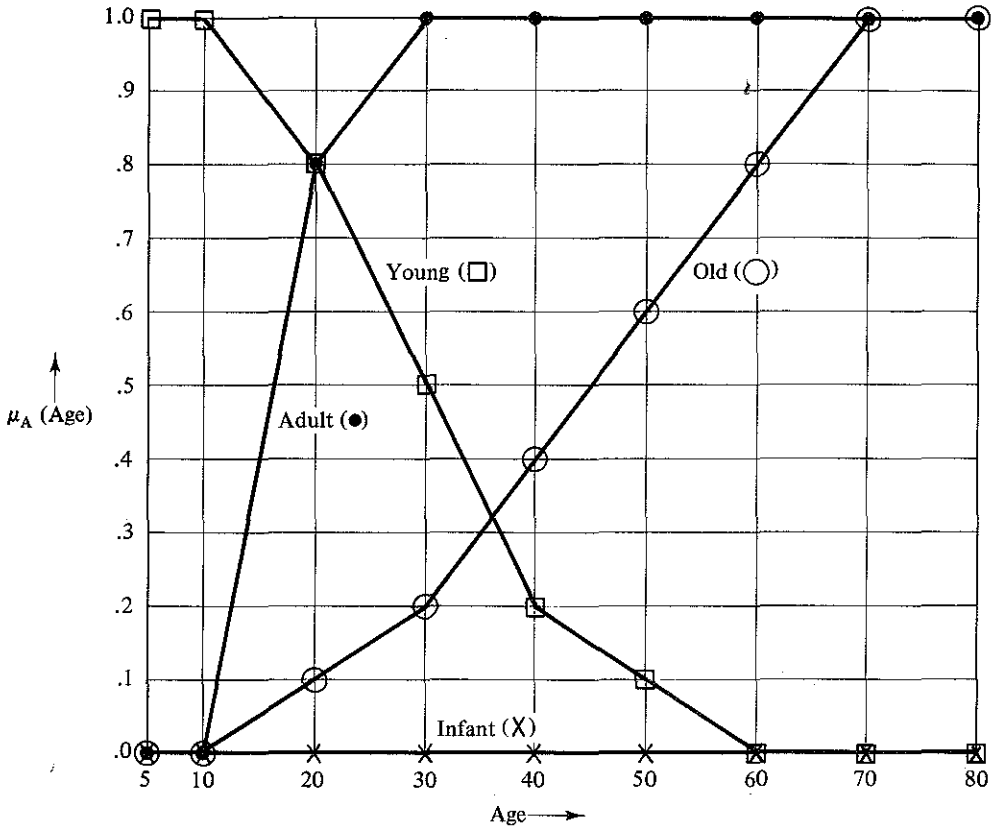


Figure 1.4. Examples of fuzzy sets defined in Table 1.2 ($A \in$ {*infant, young, adult, old*}).

is, supports of fuzzy sets in X are obtained by the function

$$\text{supp}: \tilde{\mathcal{P}}(X) \rightarrow \mathcal{P}(X),$$

where

$$\text{supp}\, A = \{x \in X \mid \mu_A(x) > 0\}.$$

For instance, the support of the fuzzy set *young* from Table 1.2 is the crisp set

$$\text{supp}(young) = \{5, 10, 20, 30, 40, 50\}.$$

An *empty fuzzy set* has an empty support; that is, the membership function assigns 0 to all elements of the universal set. The fuzzy set *infant* as defined in Table 1.2 is one example of an empty fuzzy set within the chosen universe.

Let us introduce a special notation that is often used in the literature for defining fuzzy sets with a finite support. Assume that x_i is an element of the support of fuzzy set A and that μ_i is its grade of membership in A. Then A is written as

$$A = \mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n,$$

where the slash is employed to link the elements of the support with their grades of membership in A and the plus sign indicates, rather than any sort of algebraic addition, that the listed pairs of elements and membership grades collectively form the definition of the set A. For the case in which a fuzzy set A is defined on a universal set that is finite and countable, we may write

$$A = \sum_{i=1}^{n} \mu_i / x_i.$$

Similarly, when X is an interval of real numbers, a fuzzy set A is often written in the form

$$A = \int_X \mu_A(x)/x.$$

The *height* of a fuzzy set is the largest membership grade attained by any element in that set. A fuzzy set is called *normalized* when at least one of its elements attains the maximum possible membership grade. If membership grades range in the closed interval between 0 and 1, for instance, then at least one element must have a membership grade of 1 for the fuzzy set to be considered normalized. Clearly, this will also imply that the height of the fuzzy set is equal to 1. The three fuzzy sets *adult*, *young*, and *old* from Table 1.2 as well as those defined by Figs. 1.2 and 1.3 are all normalized, and thus the height of each is equal to 1. Figure 1.5 illustrates a fuzzy set that is not normalized.

An α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a membership grade in A greater than or equal to

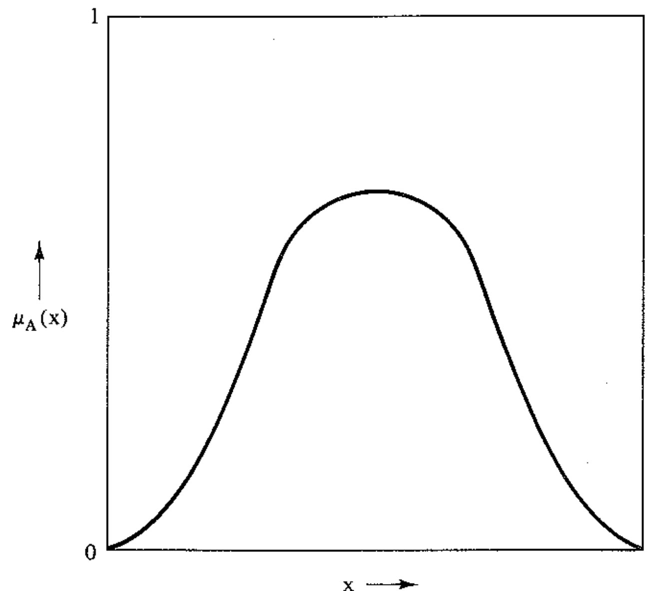


Figure 1.5. Nonnormalized fuzzy set that is convex.

the specified value of α. This definition can be written as

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

The value α can be chosen arbitrarily but is often designated at the values of the membership grades appearing in the fuzzy set under consideration. For instance, for $\alpha = .2$, the α-cut of the fuzzy set *young* from Table 1.2 is the crisp set

$$young_{.2} = \{5, 10, 20, 30, 40\}.$$

For $\alpha = .8$,

$$young_{.8} = \{5, 10, 20\},$$

and for $\alpha = 1$,

$$young_1 = \{5, 10\}.$$

Observe that the set of all α-cuts of any fuzzy set on X is a family of nested crisp subsets of X.

The set of all levels $\alpha \in [0, 1]$ that represent distinct α-cuts of a given fuzzy set A is called a *level set* of A. Formally,

$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha \text{ for some } x \in X\},$$

where Λ_A denotes the level set of fuzzy set A defined on X.

When the universal set is the set of all n-tuples of real numbers in the n-dimensional Euclidean vector space $\mathbb{R}^n$, the concept of set convexity can be generalized to fuzzy sets. A fuzzy set is *convex* if and only if each of its α-cuts is a convex set. Equivalently we may say that a fuzzy set A is convex if and only if

$$\mu_A(\lambda\mathbf{r} + (1 - \lambda)\mathbf{s}) \geq \min[\mu_A(\mathbf{r}), \mu_A(\mathbf{s})],$$

for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and all $\lambda \in [0, 1]$. Figures 1.2, 1.4, and 1.5 illustrate convex fuzzy sets, whereas Fig. 1.6 illustrates a nonconvex fuzzy set on $\mathbb{R}$. Figure 1.7 illustrates a convex fuzzy set on $\mathbb{R}^2$ expressed by the α-cuts for all α in its level set. Note that the definition of convexity for fuzzy sets does not mean that the membership function of a convex fuzzy set is necessarily a convex function.

A convex and normalized fuzzy set defined on $\mathbb{R}$ whose membership function is piecewise continuous is called a *fuzzy number*. Thus, a fuzzy number can be thought of as containing the real numbers within some interval to varying degrees. For example, the membership function given in Fig. 1.2 can be viewed as a representation of a fuzzy number.

The *scalar cardinality* of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A. Thus,

$$|A| = \sum_{x \in X} \mu_A(x).$$

The scalar cardinality of the fuzzy set *old* from Table 1.2 is

$$|old| = 0 + 0 + .1 + .2 + .4 + .6 + .8 + 1 + 1 = 4.1$$

The scalar cardinality of the fuzzy set *infant* is 0.

Other forms of cardinality have been proposed for fuzzy sets. One of these, which is called the *fuzzy cardinality,* is defined as a fuzzy number rather than as a real number, as is the case for the scalar cardinality. When a fuzzy set A has a finite support, its fuzzy cardinality $|\tilde{A}|$ is a fuzzy set (fuzzy number) defined on $\mathbb{N}$ whose membership function is defined by

$$\mu_{|\tilde{A}|}(|A_\alpha|) = \alpha,$$

for all α in the level set of A ($\alpha \in \Lambda_A$). The fuzzy cardinality of the fuzzy set *old* from Table 1.2 is

$$|\widetilde{old}| = .1/7 + .2/6 + .4/5 + .6/4 + .8/3 + 1/2.$$

There are many ways of extending the set inclusion as well as the basic crisp set operations for application to fuzzy sets. Several of these are examined in detail in Chap. 2. The discussion here is a brief introduction to the simple definitions of set inclusion and complement and to the union and intersection operations that were first proposed for fuzzy sets.

If the membership grade of each element of the universal set X in fuzzy set A is less than or equal its membership grade in fuzzy set B, then A is called a *subset* of B. Thus, if

$$\mu_A(x) \leq \mu_B(x),$$

for every $x \in X$, then

$$A \subseteq B.$$

The fuzzy set *old* from Table 1.2 is a subset of the fuzzy set *adult* since for each element in our universal set

$$\mu_{old}(x) \leq \mu_{adult}(x).$$

Fuzzy sets A and B are called *equal* if $\mu_A(x) = \mu_B(x)$ for every element $x \in X$. This is denoted by

$$A = B.$$

Clearly, if $A = B$, then $A \subseteq B$ and $B \subseteq A$.

If fuzzy sets A and B are not equal ($\mu_A(x) \neq \mu_B(x)$ for at least one $x \in X$), we write

$$A \neq B.$$

None of the four fuzzy sets defined in Table 1.2 is equal to any of the others.

Fuzzy set A is called a *proper subset* of fuzzy set B when A is a subset of B and the two sets are not equal; that is, $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$ and $\mu_A(x) < \mu_B(x)$ for at least one $x \in X$. We can denote this by writing

$$A \subset B \quad \text{if and only if} \quad A \subseteq B \text{ and } A \neq B.$$

It was mentioned that the fuzzy set *old* from Table 1.2 is a subset of the fuzzy set *adult* and that these two fuzzy sets are not equal. Therefore, *old* can be said to be a proper subset of *adult*.

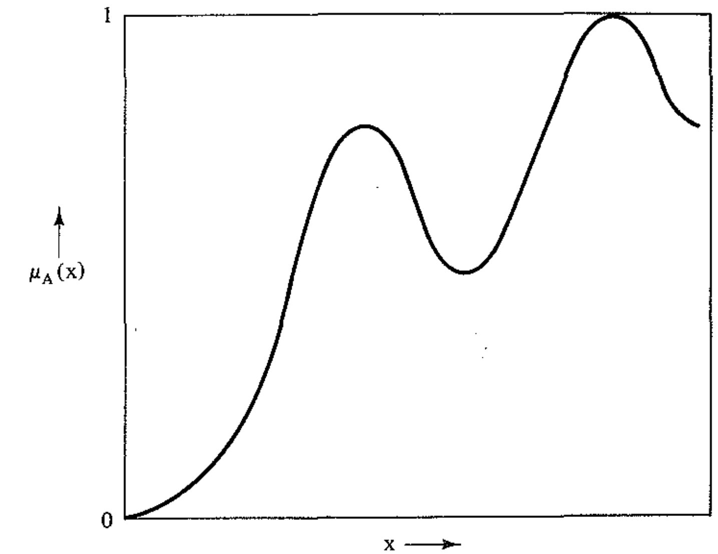


Figure 1.6. Nonconvex fuzzy set.

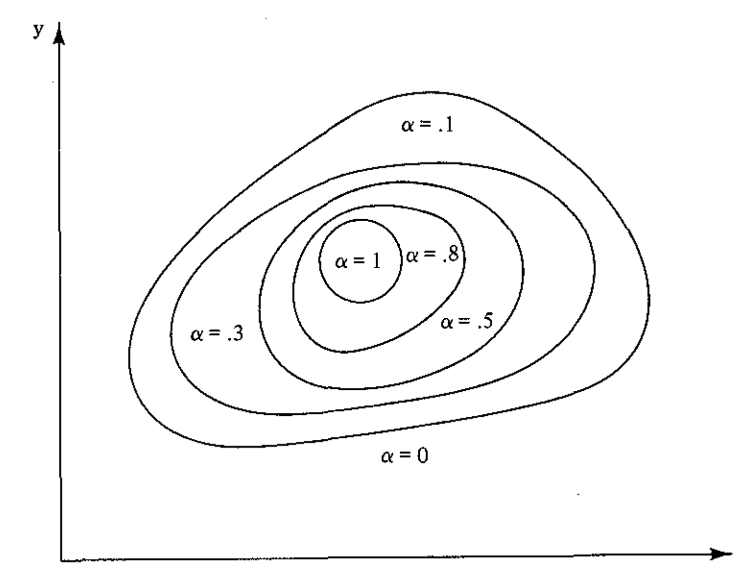


Figure 1.7. α-cuts of a convex fuzzy set defined on $\mathbb{R}^2$.

When membership grades range in the closed interval between 0 and 1, we denote the *complement* of a fuzzy set with respect to the universal set X by $\bar{A}$ and define it by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x),$$

for every $x \in X$. Thus, if an element has a membership grade of .8 in a fuzzy set A, its membership grade in the complement of A will be .2. For instance, taking the complement of the fuzzy set *old* from Table 1.2 produces the fuzzy set *not old* defined as

$$\textit{not old} = 1/5 + 1/10 + .9/20 + .8/30 + .6/40 + .4/50 + .2/60.$$

Note that in this particular case *not old* is not equal to the fuzzy set *young*.

The *union* of two fuzzy sets A and B is a fuzzy set $A \cup B$ such that

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)],$$

for every $x \in X$. Thus, the membership grade of each element of the universal set in $A \cup B$ is either its membership grade in A or its membership grade in B, whichever is the larger value. From this definition it can be seen that fuzzy sets A and B are both subsets of the fuzzy set $A \cup B$, a property we would in fact expect from a union operation. When we take the union of the fuzzy sets *young* and *old* from Table 1.2, the following fuzzy set is created:

$$\begin{aligned}\textit{young} \cup \textit{old} = {} & 1/5 + 1/10 + .8/20 + .5/30 + .4/40 \\ & + .6/50 + .8/60 + 1/70 + 1/80.\end{aligned}$$

The *intersection* of fuzzy sets A and B is a fuzzy set $A \cap B$ such that

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)],$$

for every $x \in X$. Here, the membership grade of an element x in fuzzy set $A \cap B$ is the smaller of its membership grades in set A and set B. As is desirable for an intersection operation, the fuzzy set $A \cap B$ is a subset of both A and B. The intersection of fuzzy sets *young* and *old* from Table 1.2 is a fuzzy set defined as

$$\textit{young} \cap \textit{old} = .1/20 + .2/30 + .2/40 + .1/50.$$

These original formulations of fuzzy complement, union, and intersection perform identically to the corresponding crisp set operators when membership grades are restricted to the values 0 and 1. They are, therefore, good generalizations of the classical crisp set operators. Chapter 2 contains a further discussion of the properties of these original operators and of their relation to the other classes of operators subsequently proposed.

A basic principle that allows the generalization of crisp mathematical concepts to the fuzzy framework is known as the *extension principle*. It provides the means for any function f that maps points $x_1, x_2, \ldots, x_n$ in the crisp set X to the crisp set Y to be generalized such that it maps fuzzy subsets of X to Y. Formally, given a function f mapping points in set X to points in set Y and any fuzzy set $A \in \tilde{P}(X)$ where

the extension principle states that

$$\begin{aligned} f(A) &= f(\mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n) \\ &= \mu_1/f(x_1) + \mu_2/f(x_2) + \cdots + \mu_n/f(x_n). \end{aligned}$$

If more than one element of X is mapped by f to the same element $y \in Y$, then the maximum of the membership grades of these elements in the fuzzy set A is chosen as the membership grade for y in $f(A)$. If no element $x \in X$ is mapped to y, then the membership grade of y in $f(A)$ is zero. Often a function f maps ordered tuples of elements of several different sets $X_1, X_2, \ldots, X_n$ such that $f(x_1, x_2, \ldots, x_n) = y$, $y \in Y$. In this case, for any arbitrary fuzzy sets $A_1, A_2, \ldots, A_n$ defined on $X_1, X_2, \ldots, X_n$, respectively, the membership grade of element y in $f(A_1, A_2, \ldots, A_n)$ is equal to the minimum of the membership grades of $x_1, x_2, \ldots, x_n$ in $A_1, A_2, \ldots, A_n$, respectively.

As a simple illustration of the use of this principle, suppose that f is a function mapping ordered pairs from $X_1 = \{a, b, c\}$ and $X_2 = \{x, y\}$ to $Y = \{p, q, r\}$. Let f be specified by the following matrix:

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} p & p \\ q & r \\ r & p \end{bmatrix} \end{array}$$

Let A_1 be a fuzzy set defined on X_1 and let A_2 be a fuzzy set defined on X_2 such that

$$A_1 = .3/a + .9/b + .5/c$$

and

$$A_2 = .5/x + 1/y.$$

The membership grades of p, q, and r in the fuzzy set $B = f(A_1, A_2) \in \tilde{P}(Y)$ can be calculated from the extension principle as follows:

$$\begin{aligned} \mu_B(p) &= \max[\min(.3, .5), \min(.3, 1), \min(.5, 1)] = .5; \\ \mu_B(q) &= \max[\min(.9, .5)] = .5; \\ \mu_B(r) &= \max[\min(.5, .5), \min(.9, 1)] = .9. \end{aligned}$$

Thus, by the extension principle

$$f(A_1, A_2) = .5/p + .5/q + .9/r.$$

1.5 CLASSICAL LOGIC: AN OVERVIEW

We assume that the reader of this book is familiar with the fundamentals of classical logic. Therefore, this section is solely intended to provide a brief overview of the basic concepts of classical logic and to introduce terminology and notation

Logic is the study of the methods and principles of *reasoning* in all its possible forms. Classical logic deals with *propositions* that are required to be either *true* or *false*. Each proposition has an opposite, which is usually called a *negation* of the proposition. A proposition and its negation are required to assume opposite truth values.

One area of logic, referred to as *propositional logic*, deals with combinations of variables that stand for arbitrary propositions. These variables are usually called *logic variables* (or propositional variables). As each variable stands for a hypothetical proposition, it may assume either of the two truth values; the variable is not committed to either truth value unless a particular proposition is substituted for it.

One of the main concerns of propositional logic is the study of rules by which new logic variables can be produced as functions of some given logic variables. It is not concerned with the internal structure of the propositions represented by the logic variables.

Assume that n logic variables $v_1, v_2, \ldots, v_n$ are given. A new logic variable can then be defined by a function that assigns a particular truth value to the new variable for each combination of truth values of the given variables. This function is usually called a *logic function*. Since n logic variables may assume 2^n prospective truth values, there are 2^{2^n} possible logic functions defining these variables. For example, all the logic functions of two variables are listed in Table 1.3, where falsity and truth are denoted by 0 and 1, respectively, and the resulting 16 logic variables are denoted by $w_1, w_2, \ldots, w_{16}$. Logic functions of one or two variables are usually called *logic operations*.

TABLE 1.3. LOGIC FUNCTIONS OF TWO VARIABLES.

v_2	1	1	0	0	Adopted name of function	Adopted Symbol	Other names used in the literature	Other symbols used in the literature
v_1	1	0	1	0				
w_1	0	0	0	0	Zero function	0	Falsum	F, $\perp$
w_2	0	0	0	1	Nor function	$v_1 \not\vee v_2$	Pierce function	$v_1 \downarrow v_2$, NOR(v_1, v_2)
w_3	0	0	1	0	Inhibition	$v_1 \not\Leftarrow v_2$	Proper inequality	$v_1 > v_2$
w_4	0	0	1	1	Negation	$\bar{v}_2$	Complement	$\neg v_2$, $\sim v_2$, v_2^0
w_5	0	1	0	0	Inhibition	$v_1 \not\Rightarrow v_2$	Proper inequality	$v_1 < v_2$
w_6	0	1	0	1	Negation	$\bar{v}_1$	Complement	$\neg v_1$, $\sim v_1$, v_1^0
w_7	0	1	1	0	Exclusive-or function	$v_1 ⓥ v_2$	Nonequivalence	$v_1 \not\equiv v_2$, $v_1 \oplus v_2$
w_8	0	1	1	1	Nand function	$v_1 \not\wedge v_2$	Sheffer stroke	$v_1 \mid v_2$, NAND(v_1, v_2)
w_9	1	0	0	0	Conjunction	$v_1 \wedge v_2$	And function	$v_1 \& v_2$, $v_1 v_2$
w_{10}	1	0	0	1	Biconditional	$v_1 \Leftrightarrow v_2$	Equivalence	$v_1 \equiv v_2$
w_{11}	1	0	1	0	Assertion	v_1	Identity	v_1^1
w_{12}	1	0	1	1	Implication	$v_1 \Leftarrow v_2$	Conditional, inequality	$v_1 \subset v_2$, $v_1 \geq v_2$
w_{13}	1	1	0	0	Assertion	v_2	Identity	v_2^1
w_{14}	1	1	0	1	Implication	$v_1 \Rightarrow v_2$	Conditional, inequality	$v_1 \supset v_2$, $v_1 \leq v_2$
w_{15}	1	1	1	0	Disjunction	$v_1 \vee v_2$	Or function	$v_1 + v_2$
w_{16}	1	1	1	1	One function	1	Verum	T, I

The key issue of propositional logic is the expression of all the logic functions of n variables ($n \in N$), the number of which grows extremely rapidly with increasing values of n, with the aid of a small number of simple logic functions. These simple functions are preferably logic operations of one or two variables, which are called *logic primitives*. It is known that this can be accomplished only with some sets of logic primitives. We say that a set of primitives is *complete* if and only if any logic function of variables $v_1, v_2, \ldots, v_n$ (for any finite n) can be composed by a finite number of these primitives.

Two of the many complete sets of primitives have been predominant in propositional logic: (1) negation, conjunction, and disjunction, and (2) negation and implication. By combining, for example, negations, conjunctions, and disjunctions (employed as primitives) in appropriate algebraic expressions, referred to as *logic formulas*, we can form any other logic function. Logic formulas are then defined recursively as follows:

1. The truth values 0 and 1 are logic formulas.
2. If v denotes a logic variable, then v and $\bar{v}$ are logic formulas.
3. If a and b denote logic formulas, then $a \wedge b$ and $a \vee b$ are also logic formulas.
4. The only logic formulas are those defined by statements 1 through 3.

Every logic formula of this type defines a logic function by composing it from the three primary functions. To define a unique function, the order in which the individual compositions are to be performed must be specified in some way. There are various ways in which this order can be specified. The most common is the usual use of parentheses, as in any other algebraic expression.

Other types of logic formulas can be defined by replacing some of the three operations in this definition with other operations or by including some additional operations. We may replace, for example, $a \wedge b$ and $a \vee b$ in the definition with $a \Rightarrow b$, or we may simply add $a \Rightarrow b$ to the definition.

While each proper logic formula represents a single logic function and the associated logic variable, different formulas may represent the same function and variable. If they do, we consider them equivalent. When logic formulas a and b are equivalent, we write $a = b$. For example,

$$(\bar{v}_1 \wedge \bar{v}_2) \vee (v_1 \wedge \bar{v}_3) \vee (v_2 \wedge v_3) = (\bar{v}_2 \wedge \bar{v}_3) \vee (\bar{v}_1 \wedge v_3) \vee (v_1 \wedge v_2),$$

as can easily be verified by evaluating each of the formulas for all eight combinations of truth values of the logic variables v_1, v_2, and v_3.

When the variable represented by a logic formula is always true regardless of the truth values assigned to the variables participating in the formula, it is called a *tautology*; when it is always false, it is called a *contradiction*. For example, when two logic formulas a and b are equivalent, then $a \Leftrightarrow b$ is a tautology, whereas the formula a ⓥ b is a contradiction. Tautologies are important for deductive reasoning, since they represent logic formulas that, due to their form, are true on logical grounds alone.

Various forms of tautologies can be used for making deductive inferences.

TABLE 1.4. PROPERTIES OF BOOLEAN ALGEBRAS.

(B1)	Idempotence	$a + a = a$ $a \cdot a = a$
(B2)	Commutativity	$a + b = b + a$ $a \cdot b = b \cdot a$
(B3)	Associativity	$(a + b) + c = a + (b + c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4)	Absorption	$a + (a \cdot b) = a$ $a \cdot (a + b) = a$
(B5)	Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ $a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6)	Universal bounds	$a + 0 = a, a + 1 = 1$ $a \cdot 1 = a, a \cdot 0 = 0$
(B7)	Complementarity	$a + \bar{a} = 1$ $a \cdot \bar{a} = 0$ $\bar{1} = 0$
(B8)	Involution	$\bar{\bar{a}} = a$
(B9)	Dualization	$\overline{a + b} = \bar{a} \cdot \bar{b}$ $\overline{a \cdot b} = \bar{a} + \bar{b}$

They are referred to as *inference rules*. Examples of some tautologies frequently used as inference rules are:

$$(a \wedge (a \Rightarrow b)) \Rightarrow b \qquad (\textit{modus ponens}),$$

$$(\bar{b} \wedge (a \Rightarrow b)) \Rightarrow \bar{a} \qquad (\textit{modus tollens}),$$

$$((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow (a \Rightarrow c) \qquad (\textit{hypothetical syllogism}).$$

Modus ponens, for instance, states that given two true propositions a and $a \Rightarrow b$ (the premises), the truth of the proposition b (the conclusion) may be inferred.

Every tautology remains a tautology when any of its variables is replaced with any arbitrary logic formula. This property is another example of a powerful rule of inference, referred to as a *rule of substitution*.

It is well established that propositional logic is isomorphic to set theory under the appropriate correspondence between components of these two mathematical systems. Furthermore, both of these systems are isomorphic to a Boolean algebra, which is a mathematical system defined by abstract (interpretation-free) entities and their axiomatic properties.

A *Boolean algebra* on a set B is defined as the quadruple

$$\mathcal{B} = (B, +, \cdot, \overline{}),$$

where the set B has at least two elements (bounds) 0 and 1; + and $\cdot$ are binary operations on B, and $\overline{}$ is a unary operation on B for which the properties listed in Table 1.4 are satisfied.* Properties (B1)–(B4) are common to all lattices. Boolean algebras are therefore lattices that are distributive (B5), bounded (B6), and complemented (B7)–(B9). This means that each Boolean algebra can also be characterized in terms of a partial ordering on a set that is defined as follows: $a \leq b$ if and only if $a \cdot b = a$ or, alternatively, if and only if $a + b = b$.

* Not all these properties are necessary for an axiomatic characterization of Boolean algebras; we present this larger set of properties in order to emphasize the relationship between Boolean algebras, set theory, and propositional logic.

The isomorphisms between Boolean algebra, set theory, and propositional logic guarantee that every theorem in any one of these theories has a counterpart in each of the other two theories. These counterparts can be obtained from one another by applying the substitutional correspondences in Table 1.5. All symbols used in this table have previously been defined in the text except for the symbol $\mathcal{F}(V)$; V denotes here the set of all combinations of truth values of given logic variables, and $\mathcal{F}(V)$ stands for the set of all logic functions defined in terms of these combinations. It is obviously required that the cardinalities of sets V and X be equal. These isomorphisms allow us, in effect, to cover all these theories by developing only one of them. We take advantage of this possibility by focusing the discussion in this book primarily on the theory of fuzzy sets rather than on fuzzy logic. For example, our study in Chap. 2 of the general operations on fuzzy sets is not repeated for operations of fuzzy logic, since the isomorphism between the two areas allows the properties of the latter to be obtained directly from the corresponding properties of fuzzy set operations.

Propositional logic is concerned only with those logic relationships that depend on the way in which propositions are composed from other propositions by logic operations. These latter propositions are treated as unanalyzed wholes. This is not adequate for many instances of deductive reasoning, for which the internal structure of propositions cannot be ignored.

Propositions are sentences expressed in some language. Each sentence representing a proposition can fundamentally be broken down into a *subject* and a *predicate*. In other words, a simple proposition can be expressed, in general, in the canonical form

$$x \text{ is } P,$$

where x is a symbol of a subject and P designates a predicate, which characterizes a property. For example, "Austria is a German-speaking country" is a proposition in which "Austria" stands for a subject (a particular country) and "a German

TABLE 1.5. CORRESPONDENCES DEFINING ISOMORPHISMS BETWEEN SET THEORY, BOOLEAN ALGEBRA, AND PROPOSITIONAL LOGIC.

Set theory	Boolean algebra	Propositional logic
$\mathcal{P}(X)$	B	$\mathcal{F}(V)$
$\cup$	$+$	$\vee$
$\cap$	$\cdot$	$\wedge$
$-$	$-$	$-$
X	1	1
$\varnothing$	0	0
$\subseteq$	$\leq$	$\Rightarrow$

speaking-country" is a predicate that characterizes a specific property, namely, the property of being a country whose inhabitants speak German. This proposition is true.

Instead of dealing with particular propositions, we may use the general form "x is P," where x now stands for any subject from a designated universe of discourse X. The predicate P then plays the role of a function defined on X, which for each value of x forms a proposition. This function is usually called a *predicate* and is denoted by $P(x)$. Clearly, a predicate becomes a proposition that is either true or false when a particular subject from X is substituted for x.

It is useful to extend the concept of a predicate in two ways. First, it is natural to extend it to more than one variable. This leads to the notion of an n-ary predicate $P(x_1, x_2, \ldots, x_n)$, which for $n = 1$ represents a property and for $n \geq 2$ an n-ary relation among subjects from designated universal sets X_i ($i \in \mathbb{N}_n$). For example,

$$x_1 \text{ is a citizen of } x_2,$$

where x_1 stands for individual persons from a designated population X_1 and x_2 stands for individual countries from a designated set X_2 of countries, is a binary predicate. Here, elements of X_2 are usually called *objects* rather than subjects. For convenience, n-ary predicates for $n = 0$ are defined as propositions in the same sense as in propositional logic.

Another way of extending the scope of a predicate is to quantify its applicability with respect to the domain of its variables. Two kinds of quantification have been predominantly used for predicates; they are referred to as existential quantification and universal quantification.

Existential quantification of a predicate $P(x)$ is expressed by the form

$$(\exists x)P(x),$$

which represents the sentence "There exists an individual x (in the universal set X of the variable x) such that x is P" (or the equivalent sentence "Some $x \in X$ are P"). The symbol $\exists$ is called an *existential quantifier*. We have the following equality:

$$(\exists x)P(x) = \bigvee_{x \in X} P(x). \tag{1.1}$$

Universal quantification of a predicate P(x) is expressed by the form

$$(\forall x)P(x),$$

which represents the sentence "For every individual x (in the designated universal set), x is P" (or the equivalent sentence "All $x \in X$ are P"). The symbol $\forall$ is called a *universal quantifier*. Clearly, the following equality holds:

$$(\forall x)P(x) = \bigwedge_{x \in X} P(x). \tag{1.2}$$

For n-ary predicates, we may use up to n quantifiers of either kind, each applying to one variable. For instance,

stands for the sentence "there exists an $x_1 \in X_1$ such that for all $x_2 \in X_2$ there exists x_3 such that $P(x_1, x_2, x_3)$." For example, if $X_1 = X_2 = X_3 = [0, 1]$ and $P(x_1, x_2, x_3)$ means $x_1 \leq x_2 \leq x_3$, then the sentence is true (assume $x_1 = 0$ and $x_3 = 1$).

The standard existential and universal quantification of predicates can be conveniently generalized by conceiving a quantifier Q applied to a predicate $P(x)$, $x \in X$, as a binary relation

$$Q \subset \{(\alpha, \beta) | \ \alpha, \beta \in \mathbb{N}, \alpha + \beta = |X|\},$$

where α and β specify the number of elements of X for which $P(x)$ is true or false, respectively. Formally,

$$\alpha = |\{x \in X \mid P(x) \text{ is true}\}| ,$$

$$\beta = |\{x \in X \mid P(x) \text{ is false}\}| .$$

For example, when Q is defined by the condition $\alpha \neq 0$, we obtain the standard existential quantifier; when $\beta = 0$, Q becomes the standard universal quantifier; when $\alpha > \beta$, we obtain the so-called plurality quantifier, expressed by the word *most*.

New predicates (quantified or not) can be produced from given predicates by logic formulas in the same way as new logic variables are produced by logic formulas in propositional logic. These formulas, which are called *predicate formulas*, are the essence of *predicate logic*.

1.6 FUZZY LOGIC

The basic assumption upon which classical logic (or two-valued logic) is based—that every proposition is either true or false—has been questioned since Aristotle. In his treatise *On Interpretation*, Aristotle discusses the problematic truth status of matters that are future-contingent. Propositions about future events, he maintains, are neither actually true nor actually false but are potentially either; hence, their truth value is undetermined, at least prior to the event.

It is now well understood that propositions whose truth status is problematic are not restricted to future events. As a consequence of the Heisenberg principle of uncertainty, for example, it is known that truth values of certain propositions in quantum mechanics are inherently indeterminate due to fundamental limitations of measurement. In order to deal with such propositions, we must relax the true-false dichotomy of classical two-valued logic by allowing a third truth value, which may be called *indeterminate*.

The classical two-valued logic can be extended into *three-valued logic* in various ways. Several three-valued logics, each with its own rationale, are now well established. It is common in these logics to denote the truth, falsity, and indeterminacy by 1, 0, and $\frac{1}{2}$, respectively. It is also common to define the negation $\bar{a}$ of a proposition a as $1 - a$; that is, $\bar{1} = 0$, $\bar{0} = 1$ and $\bar{\frac{1}{2}} = \frac{1}{2}$. Other primitives, three-valued logics, labeled with the names of their originators, are defined in terms of these four primitives in Table 1.6

We can see from Table 1.6 that all the logic primitives listed for the five three-valued logics fully conform to the usual definitions of these primitives in the classical logic for $a, b \in \{0, 1\}$ and that they differ from each other only in their treatment of the new truth value $\frac{1}{2}$. We can also easily verify that none of these three-valued logics satisfies the law of contradiction ($a \wedge \bar{a} = 0$), the law of excluded middle ($a \vee \bar{a} = 1$), and some other tautologies of two-valued logic. The Bochvar three-valued logic, for example, clearly does not satisfy any of the tautologies of two-valued logic, since each of its primitives produces the truth value $\frac{1}{2}$ whenever at least one of the propositions a and b assumes this value. It is, therefore, common to extend the usual concept of a tautology to the broader concept of a *quasi-tautology*. We say that a logic formula in a three-valued logic that does not assume the truth value 0 (falsity) regardless of the truth values assigned to its proposition variables is a quasi-tautology. Similarly, we say that a logic formula that does not assume the truth value 1 (truth) is a *quasi-contradiction*.

Once the various three-valued logics were accepted as meaningful and useful, it became desirable to explore generalizations into *n-valued logics* for an arbitrary number of truth values ($n \geq 2$). Several n-valued logics were, in fact, developed in the 1930s. For any given n, the truth values in these generalized logics are usually labeled by rational numbers in the unit interval [0, 1]. These values are obtained by evenly dividing the interval between 0 and 1, exclusive. The set T_n of truth values of an n-valued logic is thus defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as *degrees of truth.*

The first series of n-valued logics for which $n \geq 2$ was proposed by Łukasiewicz in the early 1930s as a generalization of his three-valued logic. It uses truth values in T_n and defines the primitives by the following equations:

$$\begin{aligned} \bar{a} &= 1 - a, \\ a \wedge b &= \min(a, b), \\ a \vee b &= \max(a, b), \\ a \Rightarrow b &= \min(1, 1 + b - a), \\ a \Leftrightarrow b &= 1 - |a - b|. \end{aligned} \tag{1.3}$$

Łukasiewicz, in fact, used only negation and implication as primitives and defined the other logic operations in terms of these two primitives, as follows:

$$\begin{aligned} a \vee b &= (a \Rightarrow b) \Rightarrow b, \\ a \wedge b &= \overline{\bar{a} \vee \bar{b}}, \end{aligned}$$

TABLE 1.6. PRIMITIVES OF SOME THREE-VALUED LOGICS.

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\Rightarrow$	$\Leftrightarrow$	$\wedge$	$\vee$	$\Rightarrow$	$\Leftrightarrow$	$\wedge$	$\vee$	$\Rightarrow$	$\Leftrightarrow$	$\wedge$	$\vee$	$\Rightarrow$	$\Leftrightarrow$	$\wedge$	$\vee$	$\Rightarrow$	$\Leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

It can be easily verified that Eqs. (1.3) become the definitions of the usual primitives of two-valued logic when $n = 2$ and that they define the primitives of Łukasiewicz's three-valued logic as given in Table 1.6.

For each $n \geq 2$, the n-valued logic of Łukasiewicz is usually denoted in the literature by L_n. The truth values of L_n are taken from T_n and its primitives are defined by Eqs. (1.3). The sequence (L_2, L_3, . . . , L_∞) of these logics contains two extreme cases—logics L_2 and L_∞. Logic L_2 is clearly the classical two-valued logic discussed in Sec. 1.5. Logic L_∞ is an *infinite-valued logic* whose truth values are taken from the set T_∞ of all rational numbers in the unit interval [0, 1].

When we do not insist on taking truth values only from the set T_∞ but rather accept as truth values any real numbers in the interval [0, 1], we obtain an alternative infinite-valued logic. Primitives of both of these infinite-valued logics are defined by Eqs. (1.3); they differ in their sets of truth values. Whereas one of these logics uses the set T_∞ as truth values, the other employs the set of all real numbers in the interval [0, 1]. In spite of this difference, these two infinite-valued logics are established as essentially equivalent in the sense that they represent exactly the same tautologies. This equivalence holds, however, only for logic formulas involving propositions; for predicate formulas with quantifiers, some fundamental differences between the two logics emerge.

Unless otherwise stated, the term *infinite-valued logic* is usually used in the literature to indicate the logic whose truth values are represented by all the real numbers in the interval [0, 1]. This is also quite often called the *standard Łukasiewicz logic* L_1, where the subscript 1 is an abbreviation for $\aleph_1$ (read "aleph 1"), which is the symbol commonly used to denote the cardinality of the continuum.

Given the isomorphism that exists between logic and set theory as defined in Table 1.5, we can see that the standard Łukasiewicz logic L_1 is isomorphic to the original fuzzy set theory based on the min, max, and $1 - a$ operators for fuzzy set intersection, union, and complement, respectively, in the same way as the two-valued logic is isomorphic to the crisp set theory. In fact, the membership grades $\mu_A(x)$ for $x \in X$ by which a fuzzy set A on the universal set X is defined can be interpreted as the truth values of the in L_1, where P is a vague (fuzzy) predicate (such as tall, young, expensive, dangerous, and so on), can be interpreted as the membership degrees $\mu_P(x)$ by which the fuzzy set characterized by the property P is defined on X. The isomorphism then follows from the fact that the logic operations of L_1, defined by Eqs. (1.3), have exactly the same mathematical form as the corresponding standard operations on fuzzy sets.

The standard Łukasiewicz logic L_1 is only one of a variety of infinite-valued logics in the same sense as the standard fuzzy set theory is only one of a variety of fuzzy set theories, which differ from one another by the set operations they employ. For each particular infinite-valued logic, we can derive the isomorphic fuzzy set theory by the correspondence in Table 1.5; a similar derivation can be made of the infinite-valued logic that is isomorphic to a given particular fuzzy set theory. A thorough study of only one of these areas, therefore, reveals the full scope of both. We are free to examine either the classes of acceptable set operations or the classes of acceptable logic operations and their various combinations. We choose in this text to focus on set operations, which are fully discussed in Chap. 2. The isomorphic logic operations and their combinations, which we do not cover explicitly, are nevertheless utilized in some of the applications discussed in Chap. 6.

The insufficiency of any single infinite-valued logic (and therefore the desirability of a variety of these logics) is connected with the notion of a complete set of logic primitives. It is known that there exists no finite complete set of logic primitives for any infinite-valued logic. Hence, using a finite set of primitives that defines an infinite-valued logic, we can obtain only a subset of all the logic functions of the given primary logic variables. Because some applications require functions outside this subset, it may become necessary to resort to alternative logics.

Since, as argued in this section, the various many-valued logics have their counterparts in fuzzy set theory, they form the kernel of *fuzzy logic*, that is, a logic based on fuzzy set theory. In its full scale, however, fuzzy logic is actually an extension of many-valued logics. Its ultimate goal is to provide foundations for *approximate reasoning* with imprecise propositions using fuzzy set theory as the principle tool. This is analogous to the role of quantified predicate logic for reasoning with precise propositions.

The primary focus of fuzzy logic is on natural language, where approximate reasoning with imprecise propositions is rather typical. The following syllogism is an example of approximate reasoning in linguistic terms that cannot be dealt with by the classical predicate logic:

Old coins are usually rare collectibles.

Rare collectibles are expensive.

Old coins are usually expensive.

This is a meaningful deductive inference. In order to deal with inferences such as this, fuzzy logic allows the use of *fuzzy predicates* (expensive, old, rare, dan-

fuzzy truth values (quite true, very true, more or less true, mostly false, and so forth), and various other kinds of *fuzzy modifiers* (such as likely, almost impossible, or extremely unlikely).

Each simple fuzzy predicate, such as

$$x \text{ is } P$$

is represented in fuzzy logic by a fuzzy set, as described previously. Assume, for example, that x stands for the *age* of a person and that P has the meaning of *young*. Then, assuming that the universal set is the set of integers from 0 to 60 representing different ages, the predicate may be represented by a fuzzy set whose membership function is shown in Fig. 1.8(a). Consider now the truth value of a proposition obtained by a particular substitution for x into the predicate, such as

Tina is young.

The truth value of this proposition depends not only on the membership grade of Tina's age in the fuzzy set chosen to characterize the concept of a young person (Fig. 1.8(a)) but also depends upon the strength of truth (or falsity) claimed. Examples of some possible truth claims are:

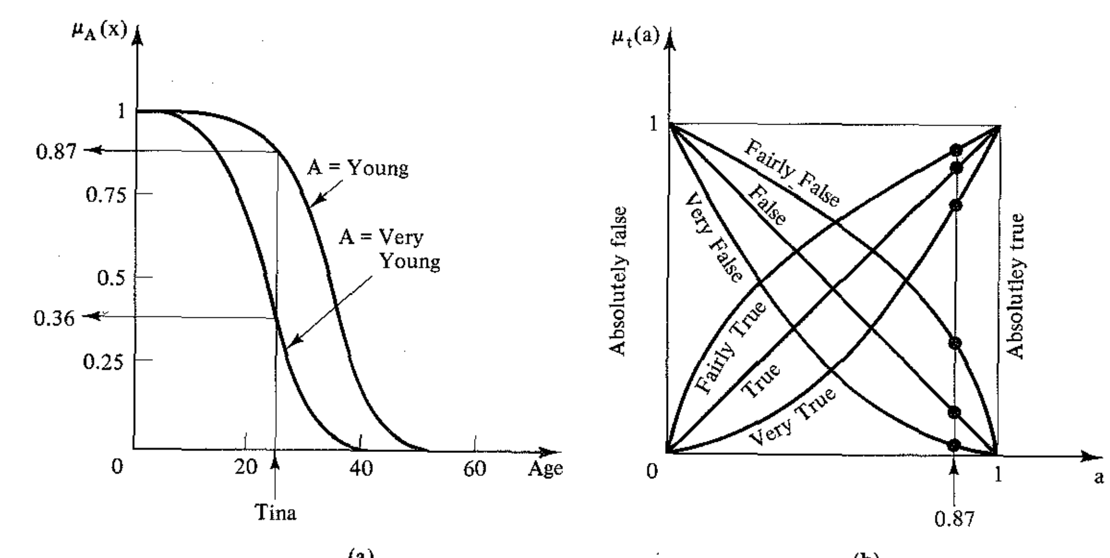
Tina is young is true.

Tina is young is false.

Tina is young is fairly true.

Tina is young is very false.

Each of the possible truth claims is represented by an appropriate fuzzy set. All



these sets are defined on the unit interval [0, 1]. Some examples are shown in Fig. 1.8(b), where a stands for the membership grade in the fuzzy set that represents the predicate involved and t is a common label representing each of the fuzzy sets in the figure that expresses truth values. Thus, in our case $a = \mu_{young}(x)$ for each $x \in X$. Returning now to Tina, who is 25 years old, we obtain $\mu_{young}(25) = .87$ (Fig. 1.8(a)), and the truth values of the propositions

Tina is young is fairly true (true, very true, fairly false,
false, very false)

are .9 (.87, .81, .18, .13, .1), respectively.

We may operate on fuzzy sets representing predicates with any of the basic fuzzy set operations of complementation, union, and intersection. Furthermore, these sets can be modified by special operations corresponding to linguistic terms such as very, extremely, more or less, quite, and so on. These terms are often called *linguistic hedges*. For example, applying the linguistic hedge *very* to the fuzzy set labeled as *young* in Fig. 1.8(a), we obtain a new fuzzy set representing the concept of a *very young* person, which is specified in the same figure.

In general, fuzzy quantifiers are represented in fuzzy logic by fuzzy numbers. These are manipulated in terms of the operations of fuzzy arithmetic, which is now well established.

From this brief outline of fuzzy logic we can see that it is operationally based on a great variety of manipulations with fuzzy sets, through which reasoning in natural language is approximated. The principles underlying these manipulations are predominantly *semantic* in nature. While full coverage of these principles is beyond the scope of this book, Chap. 6 contains illustrations of some aspects of fuzzy reasoning in the context of a few specific applications.

NOTES

1.1. The *theory of fuzzy sets* was founded by Lotfi Zadeh [1965a], primarily in the context of his interest in the analysis of complex systems [Zadeh, 1962, 1965b, 1973]. However, some of the key ideas of the theory were envisioned by Max Black, a philosopher, almost 30 years prior to Zadeh's seminal paper [Black, 1937].

1.2. The development of fuzzy set theory since its introduction in 1965 has been dramatic. Thousands of publications are now available in this new area. A survey of the status of the theory and its applications in the late 1970s is well covered in a book by Dubois and Prade [1980a]. Current contributions to the theory are scattered in many journals and books of collected papers, but the most important source is the specialized journal *Fuzzy Sets and Systems* (North-Holland). A very comprehensive bibliography of fuzzy set theory appears in a book by Kandel [1982]. An excellent annotated bibliography covering the first decade of fuzzy set theory was prepared by Gaines and Kohout [1977]. Books by Kaufmann [1975], Zimmermann [1985], and Kandel [1986] are useful supplementary readings on fuzzy set theory.

... introduced by Goguen [1967]. A thorough inves-

[1971b], was investigated by Gottwald [1979]. Convex fuzzy sets were studied in greater detail by Lowen [1980] and Liu [1985].

1.4. One concept that is only mentioned in this book but not sufficiently developed is the concept of a *fuzzy number*. It is a basis for *fuzzy arithmetic*, which can be viewed as an extension of interval arithmetic [Moore, 1966, 1979]. Among other applications, fuzzy numbers are essential for expressing *fuzzy cardinalities* and, consequently, *fuzzy quantifiers* [Dubois and Prade, 1985c]. Fuzzy arithmetic is thus a basic tool for dealing with fuzzy quantifiers in approximate reasoning; it is also a basis for developing a *fuzzy calculus* [Dubois and Prade, 1982b]. We do not cover fuzzy arithmetic, since there now exists an excellent book devoted solely to this subject [Kaufmann and Gupta, 1985].

1.5. The *extension principle* was introduced by Zadeh [1975b]. A further elaboration of the principle was presented by Yager [1986].

1.6. Fuzzy extensions of some mathematical subject areas are beyond the scope of this introductory text and are thus not covered here. They include, for example, *fuzzy topological spaces* [Chang, 1968; Wong, 1975; Lowen, 1976], *fuzzy metric spaces* [Kaleva and Seikkala, 1984], and *fuzzy games* [Butnariu, 1978].

1.7. An excellent and comprehensive survey of many-valued logics was prepared by Rescher [1969]; it also contains an extensive bibliography on the subject. Various aspects of the relationship between many-valued logics and fuzzy logic are examined by numerous authors, including Baldwin [1979a, b, c], Baldwin and Guild [1980a, b], Baldwin and Pilsworth [1980], Dubois and Prade [1979a, 1984a], Gaines [1976, 1978, 1983], Giles [1977], Gottwald [1980], Lee and Chang [1971], Mizumoto [1981], Skala [1978], Turksen and Yao [1984], and White [1979]. Approximate reasoning based on fuzzy predicate logic is also investigated in some of these papers. Particularly good overview papers were prepared by Zadeh [1975c, 1984, 1985] and Gaines [1976]. Most aspects of approximate reasoning were developed by Zadeh [1971b, 1972, 1975b, c, 1976, 1978b, 1983a, b, 1984, 1985], but we should also mention an early and important paper by Goguen [1968–69].

1.8. An alternative set theory, which is referred to as the *theory of semisets*, was proposed and developed by Vopěnka and Hájek [1972] to represent sets with imprecise boundaries. Unlike fuzzy sets, however, semisets may be defined in terms of vague properties and not necessarily by explicit membership grade functions. While semisets are more general than fuzzy sets, they are required to be approximated by fuzzy sets in practical situations. The relationship between semisets and fuzzy sets is well characterized by Novák [1984]. The concept of semisets leads into a formulation of an *alternative (nonstandard) set theory* [Vopěnka, 1979].

1.9. For a general background on crisp sets and classical two-valued logic, we recommend the book *Set Theory and Related Topics* by S. Lipschutz (Shaum, New York, 1964). The book covers all topics that are needed for this text and contains many solved examples. For a more advanced treatment of the topics, we recommend the book *Set Theory and Logic* by R. R. Stoll (W.H. Freeman, San Francisco, 1961).

EXERCISES

1.1. For each of the properties of crisp set operations listed in Table 1.1, determine

1.2. Compute the scalar cardinality and the fuzzy cardinality for each of the following fuzzy sets:
(a) $A = .4/v + .2/w + .5/x + .4/y + 1/z$;
(b) $B = 1/x + 1/y + 1/z$;
(c) $\mu_C(x) = \dfrac{x}{x+1}$, $x \in \{0, 1, 2, \ldots, 10\}$.

1.3. Consider the fuzzy sets A, B, and C defined on the interval $X = [0, 10]$ of real numbers by the membership grade functions

$$\mu_A(x) = \frac{x}{x+2}, \qquad \mu_B(x) = 2^{-x}, \qquad \mu_C(x) = \frac{1}{1 + 10(x-2)^2}.$$

Determine mathematical formulas and graphs of the membership grade functions of each of the following:
(a) $\bar{A}, \bar{B}, \bar{C}$;
(b) $A \cup B, A \cup C, B \cup C$;
(c) $A \cap B, A \cap C, B \cap C$;
(d) $A \cup B \cup C, A \cap B \cap C$;
(e) $A \cap \bar{C}, \overline{B \cap C}, \overline{A \cup C}$.

1.4. Show that DeMorgan's laws are satisfied for the three pairs of fuzzy sets obtained from fuzzy sets A, B, and C in Exercise 1.6.

1.5. Propose an extension of the standard fuzzy set operations (min, max, $1 - a$) to interval-valued fuzzy sets.

1.6. Order the fuzzy sets defined by the following membership grade functions (assuming $x \geq 0$) by the inclusion (subset) relation:

$$\mu_A(x) = \frac{1}{1 + 20x}, \qquad \mu_B(x) = \left(\frac{1}{1 + 10x}\right)^{1/2}, \qquad \mu_C(x) = \left(\frac{1}{1 + 10x}\right)^{2}.$$

1.7. Let the membership grade functions of sets A, B, and C in Exercise 1.3 be defined on the set $X = \{0, 1, \ldots, 10\}$ and let $f(x) = x^2$ for all $x \in X$. Use the extension principle to derive $f(A)$, $f(B)$, and $f(C)$.

1.8. Define α-cuts of each of the fuzzy sets defined in Exercises 1.2 and 1.3 for $\alpha = .2, .5, .9, 1$.

1.9. Show that all α-cuts of any fuzzy set A defined on $\mathbb{R}^n$ ($n \geq 1$) are convex if and only if

$$\mu_A[\lambda \mathbf{r} + (1 - \lambda)\mathbf{s}] \geq \min[\mu_A(\mathbf{r}), \mu_A(\mathbf{s})]$$

for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and all $\lambda \in [0, 1]$.

1.10. For each of the three-valued logics defined in Table 1.6, determine the truth values of each of the following logic expressions for all combinations of truth values of logic variables a, b, c (assume that negation $\bar{a}$ is defined by $1 - a$):
(a) $(\bar{a} \wedge b) \Rightarrow c$;
(b) $(\bar{a} \vee \bar{b}) \Leftrightarrow \overline{(a \wedge b)}$;
(c) $(a \Rightarrow b) \Rightarrow (\bar{c} \Rightarrow a)$.

1.11. Define in the form of a table (analogous to Table 1.6) primitives, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$, of the Łukasiewicz logics L_4 and L_5.

1.12. Repeat the example illustrated by Fig. 1.8, which is discussed in Sec. 1.6 (Tina is

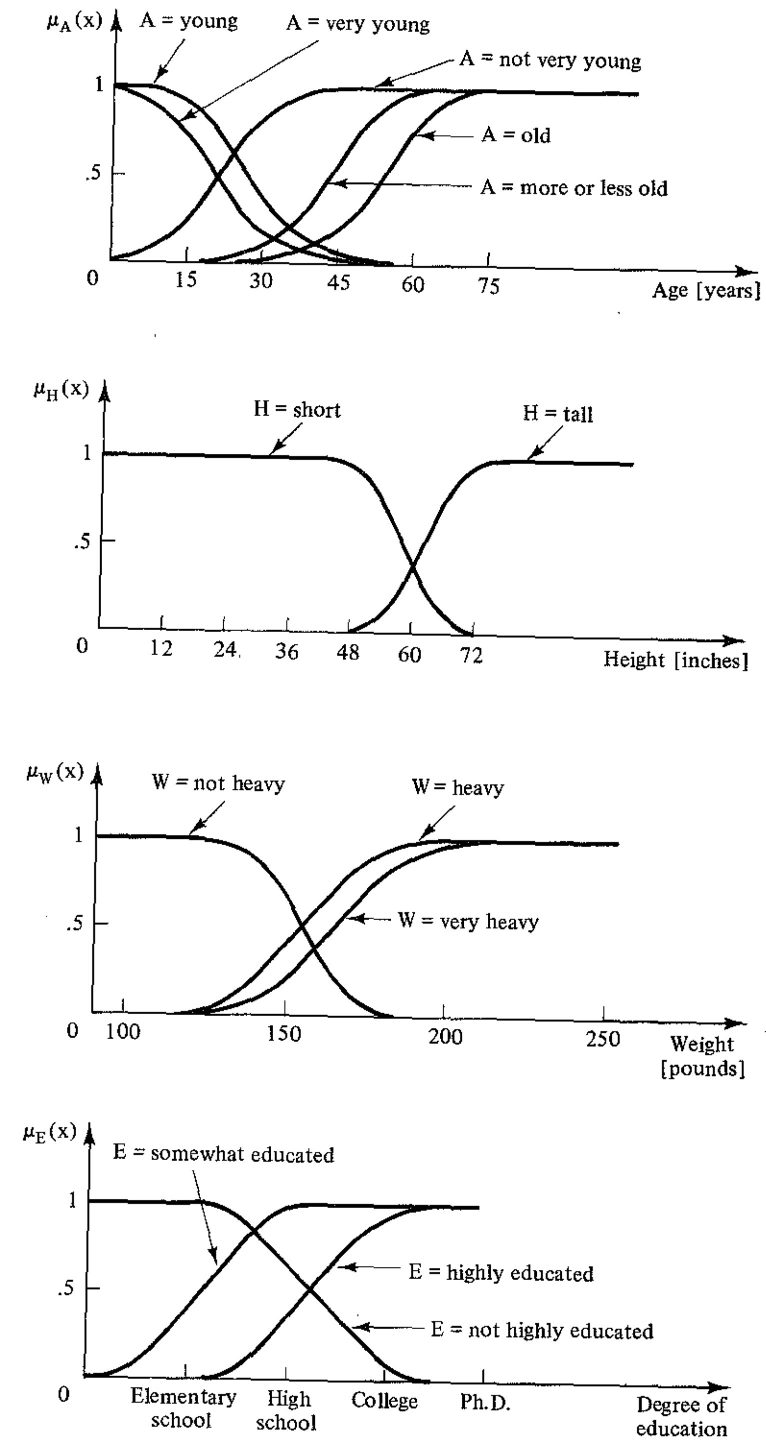


Figure 1.9. Fuzzy sets for Exercise 1.13.

1.13. Assume four types of fuzzy predicates applicable to persons (age, height, weight, and level of education). Several specific fuzzy predicates for each of these types are represented by fuzzy sets whose membership functions are specified in Fig. 1.9. Apply these membership functions and the fuzzy truth values defined in Fig. 1.8(b) to some person x (perhaps yourself) to determine the truth values of various propositions such as the following:

x is highly educated and not very young is very true;
x is very young, tall, not heavy, and somewhat educated is true;
x is more or less old or highly educated is fairly true;
x is very heavy or old or not highly educated is fairly true;
x is short, not very young and highly educated is very true.

In your calculations, use standard fuzzy set operators (min, max, $1 - a$).

2

OPERATIONS ON FUZZY SETS

2.1 GENERAL DISCUSSION

As mentioned in Chap. 1, the original theory of fuzzy sets was formulated in terms of the following specific operators of set complement, union, and intersection:

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \tag{2.1}$$

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)], \tag{2.2}$$

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]. \tag{2.3}$$

Note that when the range of membership grades is restricted to the set {0, 1}, these functions perform precisely as the corresponding operators for crisp sets, thus establishing them as clear generalizations of the latter. It is now understood, however, that these functions are not the only possible generalizations of the crisp set operators. For each of the three set operations, several different classes of functions, which possess appropriate axiomatic properties, have subsequently been proposed. This chapter contains discussions of these desirable properties and defines some of the different classes of functions satisfying them.

Despite this variety of fuzzy set operators, however, the original complement, union, and intersection still possess particular significance. Each defines a special case within all the various classes of satisfactory functions. For instance, if the functions within a class are interpreted as performing union or intersection operations of various strengths, then the classical max union is found to be the strongest of these and the classical min intersection, the weakest. Furthermore, a particularly desirable feature of these original operators is their inherent prevention of the compounding of errors of the operands. If any error e is associated with the membership grade of x in $\overline{A}$, $A \cup B$, or $A \cap B$ remains e. Many of the alternative fuzzy set operator functions later proposed lack this characteristic.

Fuzzy set theory that is based on the operators given by Eqs. (2.1) through (2.3) is now usually referred to as *possibility theory*. This theory emerges, quite naturally, as a special case of fuzzy measures. It is covered in this latter context in Chap. 4. For convenience, let the operations defined by Eqs. (2.1) through (2.3) be called the *standard operations* of fuzzy set theory.

2.2 FUZZY COMPLEMENT

A complement of a fuzzy set A is specified by a function

$$c : [0, 1] \rightarrow [0, 1],$$

which assigns a value $c(\mu_A(x))$ to each membership grade $\mu_A(x)$. This assigned value is interpreted as the membership grade of the element x in the fuzzy set representing the negation of the concept represented by A. Thus, if A is the fuzzy set of tall men, its complement is the fuzzy set of men who are not tall. Obviously, there are many elements that can have some nonzero degree of membership in both a fuzzy set and in its complement.

In order for any function to be considered a fuzzy complement, it must satisfy at least the following two axiomatic requirements:

Axiom c1. $c(0) = 1$ and $c(1) = 0$, that is, c behaves as the ordinary complement for crisp sets (*boundary conditions*).

Axiom c2. For all $a, b \in [0, 1]$, if $a < b$, then $c(a) \geq c(b)$, that is, c is *monotonic nonincreasing*.

Symbols a and b, which are used in Axiom c2 and the rest of this section as arguments of the function c, represent degrees of membership of some arbitrary elements of the universal set in a given fuzzy set. For example, $a = \mu_A(x)$ and $b = \mu_A(y)$ for some $x, y \in X$ and some fuzzy set A.

There are many functions satisfying Axioms c1 and c2. For any particular fuzzy set A, different fuzzy sets can be said to constitute its complement, each being produced by a different fuzzy complement function. In order to distinguish the complement resulting from the application of the classical fuzzy complement of Eq. (2.1) and these numerous others, the former is denoted in this text by $\overline{A}$; the latter, expressed by function c, is denoted by $C(A)$, where

$$C : \tilde{\mathcal{P}}(X) \rightarrow \tilde{\mathcal{P}}(X)$$

is a function such that $c(\mu_A(x)) = \mu_{C(A)}(x)$ for all $x \in X$.

Given a particular fuzzy complement c, function C may conveniently be used as a global operator representing c. Each function C transforms a fuzzy set A into its complement $C(A)$ as determined by the corresponding function c, which

assigns to elements of X membership grades in the complement $C(A)$. Thus each fuzzy complement c implies a corresponding function C.

All functions that satisfy Axioms c1 and c2 form the most general class of fuzzy complements. It is rather obvious that the exclusion or weakening of either of these axioms would add to this class some functions totally unacceptable as complements. Indeed, a violation of Axiom c1 would include functions that do not conform to the ordinary complement for crisp sets. Axiom c2 is essential since we intuitively expect that an increase in the degree of membership in a fuzzy set must result either in a decrease or, in the extreme case, in no change in the degree of membership in its complement. Let Axioms c1 and c2 be called the *axiomatic skeleton for fuzzy complements*.

In most cases of practical significance, it is desirable to consider various additional requirements for fuzzy complements. Each of them reduces the general class of fuzzy complements to a special subclass. Two of the most desirable requirements, which are usually listed in the literature among axioms of fuzzy complements, are the following:

Axiom c3. c is a *continuous* function.

Axiom c4. c is *involutive*, which means that $c(c(a)) = a$ for all $a \in [0, 1]$.

Functions that satisfy Axiom c3 form a special subclass of the general class of fuzzy complements; those satisfying Axiom c4 are necessarily continuous as well and, therefore, form a further nested subclass, as illustrated in Fig. 2.1. The classical fuzzy complement given by Eq. (2.1) is contained within the class of involutive complements.

Examples of general fuzzy complements that satisfy only the axiomatic skeleton are the threshold-type complements defined by

$$c(a) = \begin{cases} 1 & \text{for } a \leq t, \\ 0 & \text{for } a > t, \end{cases}$$

where $a \in [0, 1]$ and $t \in [0, 1)$; t is called the threshold of c. This function is illustrated in Fig. 2.2(a).

An example of a fuzzy complement that is continuous (Axiom c3) but not involutive (Axiom c4) is the function

$$c(a) = \tfrac{1}{2}(1 + \cos \pi a),$$

which is illustrated in Fig. 2.2(b). The failure of this function to satisfy the property of involution can be seen by noting that, for example, $c(.33) = .75$ but $c(.75) = .15 \neq .33$.

One class of involutive fuzzy complements is the *Sugeno class* defined by

$$c_\lambda(a) = \frac{1 - a}{1 + \lambda a},$$

where $\lambda \in (-1, \infty)$. For each value of the parameter λ, we obtain one particular

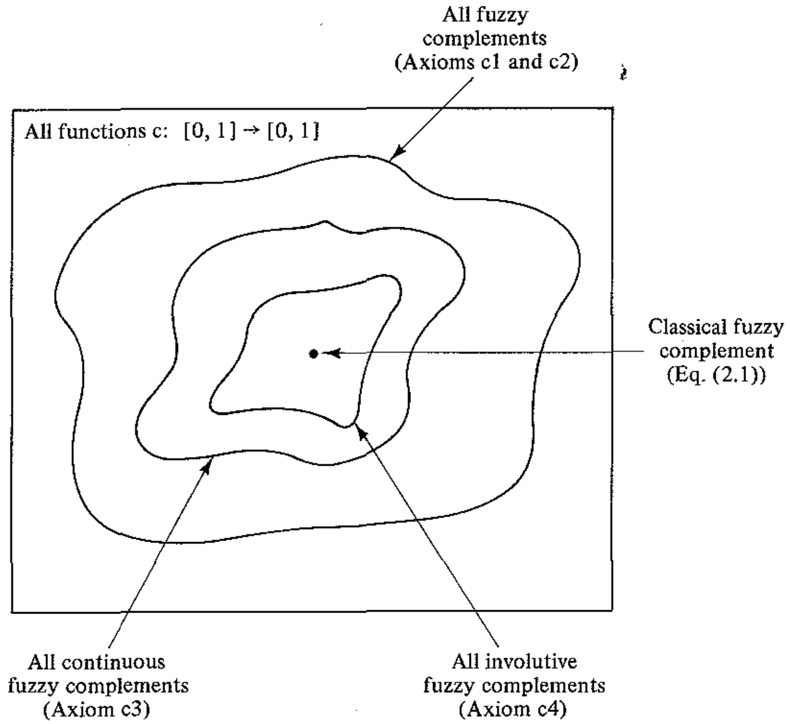


Figure 2.1. Illustration of the nested subset relationship of the basic classes of fuzzy complements.

involutive fuzzy complement. This class is illustrated in Fig. 2.3(a) for several different values of λ. Note how the shape of the function is affected as the value of λ is changed. For $\lambda = 0$, the function becomes the classical fuzzy complement defined by Eq. (2.1).

Another example of a class of involutive fuzzy complements is defined by

$$c_w(a) = (1 - a^w)^{1/w},$$

where $w \in (0, \infty)$; let us refer to it as the *Yager class* of fuzzy complements. Figure 2.3(b) illustrates this class of functions for various values of w. Here again, changing the value of the parameter w results in a deformation of the shape of the function. When $w = 1$, this function becomes the classical fuzzy complement of $c(a) = 1 - a$.

Several important properties are shared by all fuzzy complements. These concern the *equilibrium* of a fuzzy complement c, which is defined as any value a for which $c(a) = a$. In other words, the equilibrium of a complement c is that degree of membership in a fuzzy set A equaling the degree of membership in the complement $C(A)$. For instance, the equilibrium value for the classical fuzzy complement given by Eq. (2.1) is .5, which is the solution of the equation $1 - a$

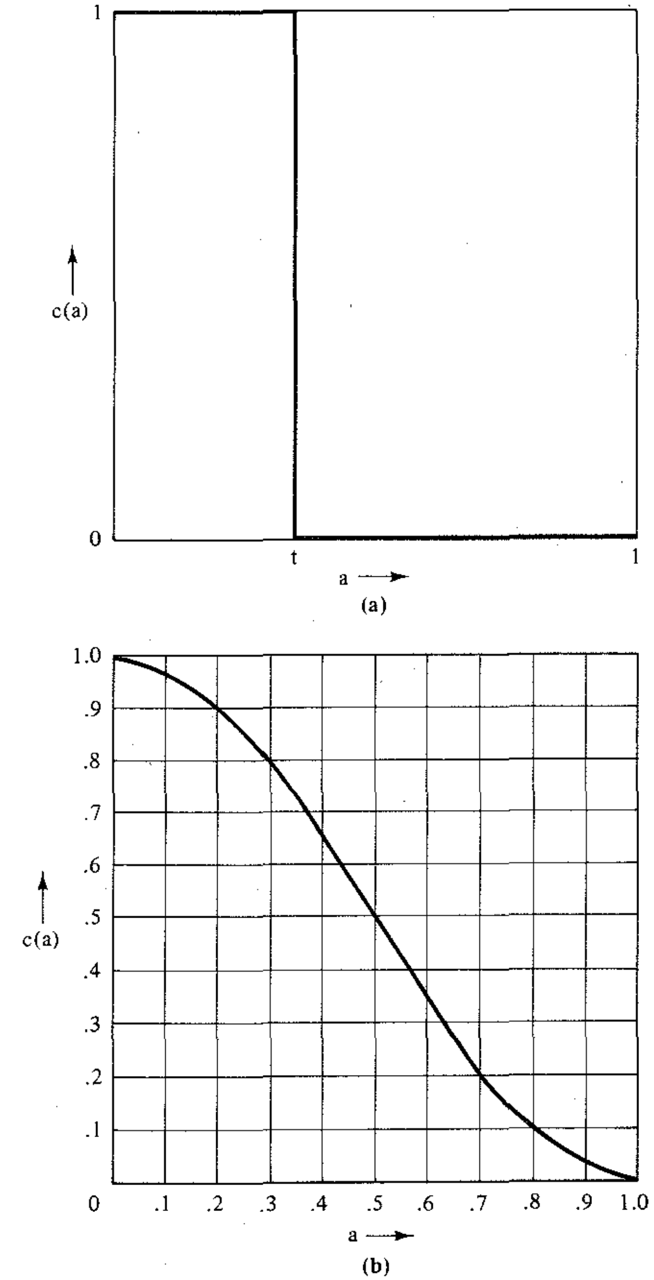


Figure 2.2. Examples of fuzzy complements: (a) a general complement of the threshold type; (b) a continuous fuzzy complement $c(a) = \frac{1}{2}(1 + \cos \pi a)$. πa).

Theorem 2.1. Every fuzzy complement has at most one equilibrium.

Proof: Let c be an arbitrary fuzzy complement. An equilibrium of c is a solution of the equation

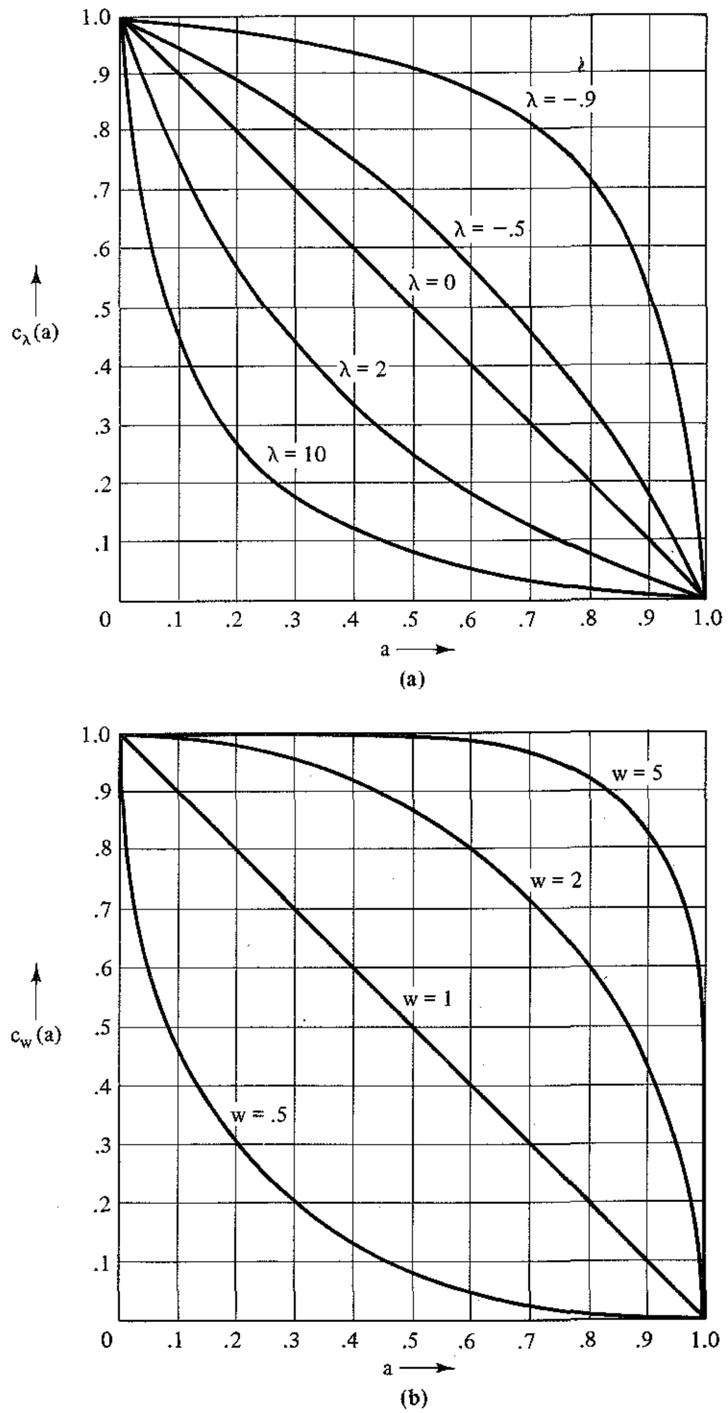


Figure 2.3. Examples from two classes of involutive fuzzy complements: (a)

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where b is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that a_1 and a_2 are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2. \tag{2.4}$$

However, because c is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts Eq. (2.4), thus demonstrating that the equation must have at most one solution. ■

Theorem 2.2. Assume that a given fuzzy complement c has an equilibrium e_c which by Theorem 2.1 is unique. Then

$$a \leq c(a) \quad \text{if and only if} \quad a \leq e_c$$

and

$$a \geq c(a) \quad \text{if and only if} \quad a \geq e_c.$$

Proof: Let us assume that $a < e_c$, $a = e_c$, and $a > e_c$, in turn. Then, since c is monotonic nonincreasing by Axiom c2, $c(a) \geq c(e_c)$ for $a < e_c$, $c(a) = c(e_c)$ for $a = e_c$, and $c(a) \leq c(e_c)$ for $a > e_c$. Because $c(e_c) = e_c$, we can rewrite these expressions as $c(a) \geq e_c$, $c(a) = e_c$, and $c(a) \leq e_c$, respectively. In fact, due to our initial assumption we can further rewrite these as $c(a) > a$, $c(a) = a$, and $c(a) < a$, respectively. Thus, $a \leq e_c$ implies $c(a) \geq a$ and $a \geq e_c$ implies $c(a) \leq a$. The inverse implications can be shown in a similar manner. ■

Theorem 2.3. If c is a continuous fuzzy complement, then c has a unique equilibrium.

Proof: The equilibrium e_c of a fuzzy complement c is the solution of the equation $c(a) - a = 0$. This is a special case of the more general equation $c(a) - a = b$, where $b \in [-1, 1]$ is a constant. By Axiom c1, $c(0) - 0 = 1$ and $c(1) - 1 = -1$. Since c is a continuous complement, it follows from the intermediate value theorem for continuous functions* that for each $b \in [-1, 1]$, there exists at least one a such that $c(a) - a = b$. This demonstrates the necessary existence of an equilibrium value for a continuous function, and Theorem 2.1 guarantees its uniqueness. ■

* See, for example, *Mathematical Analysis* (second ed.), by T. M. Apostol, Addison-Wesley,

The equilibrium for each individual fuzzy complement c_λ of the Sugeno class is given by

$$e_{c_\lambda} = \begin{cases} \dfrac{\sqrt{1+\lambda}-1}{\lambda} & \text{for } \lambda \neq 0, \\ \dfrac{1}{2} & \text{for } \lambda = 0 \left(= \lim\limits_{\lambda \to 0} \dfrac{\sqrt{1+\lambda}-1}{\lambda}\right). \end{cases}$$

This is clearly obtained by selecting the positive solution of the equation

$$\frac{1-e_{c_\lambda}}{1+\lambda e_{c_\lambda}} = e_{c_\lambda}.$$

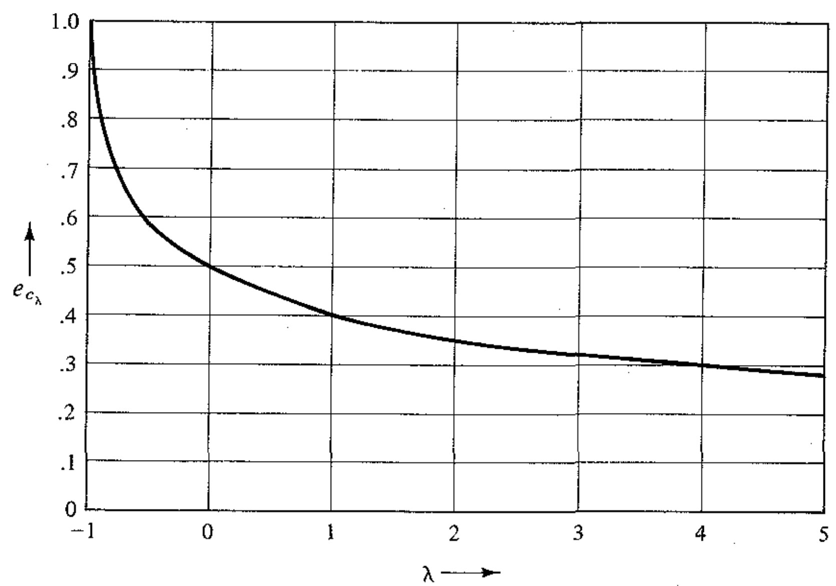
The dependence of the equilibrium e_{c_λ} on the parameter λ is shown in Fig. 2.4.

If we are given a fuzzy complement c and a membership grade whose value is represented by a real number $a \in [0, 1]$, then any membership grade represented by the real number ${}^d a \in [0, 1]$ such that

$$c({}^d a) - {}^d a = a - c(a), \tag{2.5}$$

is called a *dual point* of a with respect to c.

It follows directly from the proof of Theorem 2.1 that Eq. (2.5) has at most one solution for ${}^d a$ given c and a. There is, therefore, at most one dual point for each particular fuzzy complement c and membership grade of value a. Moreover, it follows from the proof of Theorem 2.3 that a dual point exists for each $a \in [0, 1]$ when c is a continuous complement.



Theorem 2.4. If a complement c has an equilibrium e_c, then

$${}^d e_c = e_c.$$

Proof: If $a = e_c$, then by our definition of equilibrium, $c(a) = a$ and thus $a - c(a) = 0$. Additionally, if ${}^d a = e_c$, then $c({}^d a) = {}^d a$ and $c({}^d a) - {}^d a = 0$. Therefore,

$$c({}^d a) - {}^d a = a - c(a).$$

This satisfies Eq. (2.5) when $a = {}^d a = e_c$. Hence, the equilibrium of any complement is its own dual point. ■

Theorem 2.5. For each $a \in [0, 1]$, ${}^d a = c(a)$ if and only if $c(c(a)) = a$, that is, when the complement is involutive.

Proof: Let ${}^d a = c(a)$. Then, substitution of $c(a)$ for ${}^d a$ in Eq. (2.5) produces

$$c(c(a)) - c(a) = a - c(a).$$

Therefore, $c(c(a)) = a$. For the reverse implication, let $c(c(a)) = a$. Then, substitution of $c(c(a))$ for a in Eq. (2.5) yields

$$c({}^d a) - {}^d a = c(c(a)) - c(a).$$

Because ${}^d a$ can be substituted for $c(a)$ everywhere in this equation to yield a tautology, ${}^d a = c(a)$. ■

Thus, the dual point of any membership grade is equal to its complemented value whenever the complement is involutive. If the complement is not involutive, then either the dual point does not exist or it does not coincide with the complement point.

These results associated with the concepts of the equilibrium and the dual point of a fuzzy complement are referenced in the discussion of measures of fuzziness contained in Chap. 5.

2.3 FUZZY UNION

The union of two fuzzy sets A and B is specified in general by a function of the form

$$u : [0, 1] \times [0, 1] \to [0, 1].$$

For each element x in the universal set, this function takes as its argument the pair consisting of the element's membership grades in set A and in set B and yields the membership grade of the element in the set constituting the union of A and B. Thus,

$$\mu_{A \cup B}(x) = u[\mu_A(x), \mu_B(x)].$$

In order for any function of this form to qualify as a fuzzy union, it must

Axiom u1. $u(0, 0) = 0$; $u(0, 1) = u(1, 0) = u(1, 1) = 1$; that is, u behaves as the classical union with crisp sets (*boundary conditions*).

Axiom u2. $u(a, b) = u(b, a)$; that is, u is *commutative*.

Axiom u3. If $a \leq a'$ and $b \leq b'$, then $u(a, b) \leq u(a', b')$; that is, u is *monotonic*.

Axiom u4. $u(u(a, b), c) = u(a, u(b, c))$; that is, u is *associative*.

Let us call this set of axioms the *axiomatic skeleton for fuzzy set unions*.

The first axiom insures that the function will define an operation that generalizes the classical crisp set union. The second axiom of commutativity (or symmetry) indicates indifference to the order in which the sets to be combined are considered. The third axiom is the natural requirement that a decrease in the degree of membership in set A or set B cannot produce an increase in the degree of membership in $A \cup B$. Finally, the fourth axiom of associativity ensures that we can take the union of any number of sets in any order of pairwise grouping desired; this axiom allows us to extend the operation of fuzzy set union to more than two sets.

It is often desirable to restrict the class of fuzzy unions by considering various additional requirements. Two of the most important requirements are expressed by the following axioms:

Axiom u5. u is a *continuous* function.

Axiom u6. $u(a, a) = a$; that is, u is *idempotent*.

The axiom of continuity prevents a situation in which a very small increase in the membership grade in either set A or set B produces a large change in the membership grade in $A \cup B$. Axiom u6 insures that the union of any set with itself yields precisely the same set.

Several classes of functions have been proposed whose individual members satisfy all the axiomatic requirements for the fuzzy union and neither, one, or both of the optional axioms. One of these classes of fuzzy unions is known as the *Yager class* and is defined by the function

$$u_w(a, b) = \min[1, (a^w + b^w)^{1/w}], \tag{2.6}$$

where values of the parameter w lie within the open interval $(0, \infty)$. This class of functions satisfies Axioms u1 through u5, but these functions are not, in general, idempotent. Special functions within this class are formed when certain values are chosen for the parameter w. For instance, for $w = 1$, the function becomes

$$u_1(a, b) = \min[1, a + b];$$

for $w = 2$, we obtain

Since it is not obvious what form the function u_w given by Eq. (2.6) takes for $w \to \infty$, we use the following theorem.

Theorem 2.6. $\lim_{w \to \infty} \min[1, (a^w + b^w)^{1/w}] = \max(a, b)$.

Proof: The theorem is obvious whenever (1) a or b equal 0, or (2) $a = b$, because the limit of $2^{1/w}$ as $w \to \infty$ equals 1. If $a \neq b$ and the min equals $(a^w + b^w)^{1/w}$, the proof reduces to the demonstration that

$$\lim_{w \to \infty} (a^w + b^w)^{1/w} = \max(a, b).$$

Let us assume, with no loss of generality, that $a < b$, and let $Q = (a^w + b^w)^{1/w}$. Then

$$\lim_{w \to \infty} \ln Q = \lim_{w \to \infty} \frac{\ln(a^w + b^w)}{w}.$$

Using l'Hospital's rule, we obtain

$$\lim_{w \to \infty} \ln Q = \lim_{w \to \infty} \frac{a^w \ln a + b^w \ln b}{a^w + b^w}$$

$$= \lim_{w \to \infty} \frac{(a/b)^w \ln a + \ln b}{(a/b)^w + 1} = \ln b.$$

Hence,

$$\lim_{w \to \infty} Q = \lim_{w \to \infty} (a^w + b^w)^{1/w} = b\ (= \max(a, b)).$$

It remains to show that the theorem is still valid when the min equals 1. In this case,

$$(a^w + b^w)^{1/w} \geq 1$$

or

$$a^w + b^w \geq 1$$

for all $w \in (0, \infty)$. When $w \to \infty$, the last inequality holds if $a = 1$ or $b = 1$ (since $a, b \in [0, 1]$). Hence, the theorem is again satisfied. ■

The various functions of the Yager class, which are defined by different choices of the parameter w, can be interpreted as performing union operations of various strengths. Table 2.1(a) illustrates how the values produced by the Yager functions for fuzzy unions decrease as the value of w increases. In fact, we may interpret the value $1/w$ as indicating the degree of interchangeability present in the union operation u_w. The notion of the set union operation corresponds to the logical OR (disjunction), in which some interchangeability between the two arguments of the statement "A or B" is assumed. Thus, the union of two fuzzy

TABLE 2.1. EXAMPLES OF FUZZY SET OPERATIONS FROM THE YAGER CLASS.

(a) Fuzzy unions

a \ b	0	.25	.5	.75	1
a = 1	1	1	1	1	1
.75	.75	1	1	1	1
.5	.5	.75	1	1	1
.25	.25	.5	.75	1	1
0	0	.25	.5	.75	1

$w = 1$ (soft)

a \ b	0	.25	.5	.75	1
a = 1	1	1	1	1	1
.75	.75	.79	.9	1	1
.5	.5	.56	.71	.9	1
.25	.25	.35	.56	.79	1
0	0	.25	.5	.75	1

$w = 2$

a \ b	0	.25	.5	.75	1
a = 1	1	1	1	1	1
.75	.75	.75	.75	.8	1
.5	.5	.5	.54	.75	1
.25	.25	.27	.5	.75	1
0	0	.25	.5	.75	1

$w = 10$

a \ b	0	.25	.5	.75	1
a = 1	1	1	1	1	1
.75	.75	.75	.75	.75	1
.5	.5	.5	.5	.75	1
.25	.25	.25	.5	.75	1
0	0	.25	.5	.75	1

$w \to \infty$ (hard)

(b) Fuzzy intersections

a \ b	0	.25	.5	.75	1
a = 1	0	.25	.5	.75	1
.75	0	0	.25	.5	.75
.5	0	0	0	.25	.5
.25	0	0	0	0	.25
0	0	0	0	0	0

$w = 1$ (strong)

a \ b	0	.25	.5	.75	1
a = 1	0	.25	.5	.75	1
.75	0	.21	.44	.65	.75
.5	0	.1	.29	.44	.5
.25	0	0	.1	.21	.25
0	0	0	0	0	0

$w = 2$

a \ b	0	.25	.5	.75	1
a = 1	0	.25	.5	.75	1
.75	0	.25	.5	.73	.75
.5	0	.25	.46	.5	.5
.25	0	.20	.25	.25	.25
0	0	0	0	0	0

$w = 10$

a \ b	0	.25	.5	.75	1
a = 1	0	.25	.5	.75	1
.75	0	.25	.5	.75	.75
.5	0	.25	.5	.5	.5
.25	0	.25	.25	.25	.25
0	0	0	0	0	0

$w \to \infty$ (weak)

fuzzy union for which $w = 1$, the membership grades in the two sets are summed to produce the membership grade in their union. Therefore, this union is very soft and indicates perfect interchangeability between the two arguments. On the other hand, the Yager function for which $w \to \infty$ (the classical fuzzy union) performs a very hard OR by selecting the largest degree of membership in either set. In this sense, then, the functions of the Yager class perform a union operation, which increases in strength as the value of the parameter w increases.

Some other proposed classes of fuzzy set unions along with the corresponding class of fuzzy set intersections are given in Table 2.2. They are identified by the names of their originators and the date of the publication in which they were introduced. While we do not deem it essential to examine all these various classes in this text, the information provided in Note 2.7 is sufficient to allow the reader to pursue such an examination.

2.4 FUZZY INTERSECTION

The discussion of fuzzy intersection closely parallels that of fuzzy union. Like fuzzy union, the general fuzzy intersection of two fuzzy sets A and B is specified by a function

$$i : [0, 1] \times [0, 1] \to [0, 1].$$

The argument to this function is the pair consisting of the membership grade of some element x in fuzzy set A and the membership grade of that same element in fuzzy set B. The function returns the membership grade of the element in the set $A \cap B$. Thus,

$$\mu_{A\cap B}(x) = i[\mu_A(x), \mu_B(x)].$$

A function of this form must satisfy the following axioms in order to be considered a fuzzy intersection:

Axiom i1. $i(1, 1) = 1$; $i(0, 1) = i(1, 0) = i(0, 0) = 0$; that is, i behaves as the classical intersection with crisp sets (*boundary conditions*).

Axiom i2. $i(a, b) = i(b, a)$; that is, i is *commutative*.

Axiom i3. If $a \leq a'$ and $b \leq b'$, then $i(a, b) \leq i(a', b')$; that is, i is *monotonic*.

Axiom i4. $i(i(a, b), c) = i(a, i(b, c))$, that is, i is *associative*.

The justification for these essential axioms (the *axiomatic skeleton for fuzzy set intersections*) is similar to that given in the previous section for the required axioms of fuzzy union.

The most important additional requirements for fuzzy set intersections, which are desirable in certain applications, are expressed by the following two

TABLE 2.2. SOME CLASSES OF FUZZY SET UNIONS AND INTERSECTIONS.

Reference	Fuzzy Unions	Fuzzy Intersections	Range of Parameter
Schweizer & Sklar [1961]	$1 - \max[0, (1 - a)^{-p} + (1 - b)^{-p} - 1)]^{1/p}$	$\max(0, a^{-p} + b^{-p} - 1)^{-1/p}$	$p \in (-\infty, \infty)$
Hamacher [1978]	$\dfrac{a + b - (2 - \gamma)ab}{1 - (1 - \gamma)ab}$	$\dfrac{ab}{\gamma + (1 - \gamma)(a + b - ab)}$	$\gamma \in (0, \infty)$
Frank [1979]	$1 - \log_s \left[1 + \dfrac{(s^{1-a} - 1)(s^{1-b} - 1)}{s - 1}\right]$	$\log_s \left[1 + \dfrac{(s^a - 1)(s^b - 1)}{s - 1}\right]$	$s \in (0, \infty)$
Yager [1980]	$\min[1, (a^w + b^w)^{1/w}]$	$1 - \min[1, (1 - a)^w + (1 - b)^w)^{1/w}]$	$w \in (0, \infty)$
Dubois & Prade [1980]	$\dfrac{a + b - ab - \min(a, b, 1 - \alpha)}{\max(1 - a, 1 - b, \alpha)}$	$\dfrac{ab}{\max(a,b,\alpha)}$	$\alpha \in (0, 1)$
Dombi [1982]	$\dfrac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{-\lambda} + \left(\frac{1}{b} - 1\right)^{-\lambda}\right]^{-1/\lambda}}$	$\dfrac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{b} - 1\right)^{\lambda}\right]^{1/\lambda}}$	$\lambda \in (0, \infty)$

Axiom i5. i is a *continuous* function.

Axiom i6. $i(a, a) = a$; that is, i is *idempotent.*

The implications of these two properties for fuzzy intersection operations are basically the same as those given in the previous section for fuzzy unions.

Some of the classes of functions that satisfy Axioms i1 through i4 are shown in Table 2.2. Let us examine one of these—the *Yager class*, which is defined by the function

$$i_w(a, b) = 1 - \min[1, ((1 - a)^w + (1 - b)^w)^{1/w}], \tag{2.7}$$

where values of the parameter w lie in the open interval $(0, \infty)$.

For each value of the parameter w, we obtain one particular fuzzy set intersection. Like the Yager class of fuzzy unions, all the functions of this class are continuous but most are not idempotent. For $w = 1$, the function of Eq. (2.7) is defined by

$$i_1(a, b) = 1 - \min[1, 2 - a - b];$$

for $w = 2$, we obtain

$$i_2(a, b) = 1 - \min[1, \sqrt{(1 - a)^2 + (1 - b)^2}];$$

similar values are obtained for other finite values of w. For $w \to \infty$, the form of the function given by Eq. (2.7) is not obvious and, therefore, we employ the following theorem.

Theorem 2.7. $\operatorname{Lim}_{w \to \infty} i_w = \lim_{w \to \infty} (1 - \min[1, ((1 - a)^w + (1 - b)^w)^{1/w}])$
$= \min(a, b)$.

Proof: From the proof of Theorem 2.6, we know that

$$\lim_{w \to \infty} \min[1, [(1 - a)^w + (1 - b)^w]^{1/w}] = \max[1 - a, 1 - b].$$

Thus, $i_\infty(a, b) = 1 - \max[1 - a, 1 - b]$. Let us assume, with no loss of generality, that $a \leq b$. Then, $1 - a \geq 1 - b$ and

$$i_\infty(a, b) = 1 - (1 - a) = a.$$

Hence, $i_\infty(a, b) = \min(a, b)$, which concludes the proof. ∎

As is the case with the functions in the Yager class of fuzzy unions, the choice of the parameter w determines the strength of the intersection operations performed by the Yager functions of Eq. (2.7). Table 2.1(b) illustrates the increasing values returned by the Yager intersections as the value of the parameter w increases. Thus, the value $1/w$ can be interpreted as the degree of strength of the intersection performed. Since the intersection is analogous to the logical AND (conjuction), it generally demands simultaneous satisfaction of the operands of A and B. The Yager intersection for which $w = 1$ returns a positive value only when the summation of the membership grades in the two sets exceeds 1. Thus, it performs a hard intersection with the strongest demand for simultaneous set membership. In contrast to this, the Yager function for which $w \to \infty$, which is the classical fuzzy set intersection, performs a soft intersection that allows the lowest degree of membership in either set to dictate the degree of membership in their intersection. In effect then, this operation shows the least demand for simultaneous set membership.

2.5 COMBINATIONS OF OPERATIONS

It is known that fuzzy set unions that satisfy the axiomatic skeleton (Axioms u1 through u4 given in Sec. 2.3) are bounded by the inequalities

$$\max(a, b) \leq u(a, b) \leq u_{\max}(a, b), \tag{2.8}$$

where

$$u_{\max}(a, b) = \begin{cases} a & \text{when } b = 0, \\ b & \text{when } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, fuzzy set intersections that satisfy Axioms i1 through i4 (given in Sec. 2.4) are bounded by the inequalities

$$i_{\min}(a, b) \leq i(a, b) \leq \min(a, b), \tag{2.9}$$

where

$$i_{\min}(a, b) = \begin{cases} a & \text{when } b = 1, \\ b & \text{when } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The inequalities $u(a, b) \geq \max(a, b)$ and $i(a, b) \leq \min(a, b)$ are often used as axioms for fuzzy unions and intersections, respectively, instead of the axioms of associativity. However, these inqualities as well as those for $u_{\max}$ and $i_{\min}$ can be derived from our axioms as shown in the following four theorems.

Theorem 2.8. For all $a, b \in [0, 1]$, $u(a, b) \geq \max(a, b)$.

Proof: Using associativity (Axiom u4), the equation

$$u(a, u(0, 0)) = u(u(a, 0), 0)$$

is valid. By applying the boundary condition $u(0, 0) = 0$ (Axiom u1), we can rewrite this equation as

$$u(a, 0) = u(u(a, 0), 0).$$

Assume now that the solution of this equation is $u(a, 0) = \alpha \neq a$. Substitution of α for $u(a, 0)$ in the equation yields $\alpha = u(\alpha, 0)$, which contradicts our as-

sumption. Hence, the only solution of the equation is $u(a, 0) = a$. Now, by monotonicity of u (Axiom u3), we have

$$u(a, b) \geq u(a, 0) = a,$$

and, by employing commutativity (Axiom u2), we also have

$$u(a, b) = u(b, a) \geq u(b, 0) = b.$$

Hence, $u(a, b) \geq \max(a, b)$. ■

Theorem 2.9. For all $a, b \in [0, 1]$, $u(a, b) \leq u_{\max}(a, b)$.

Proof: When $b = 0$, then $u(a, b) = a$ (see the proof of Theorem 2.8) and the theorem holds. Similarly, by commutativity, when $a = 0$, then $u(a, b) = b$, and the theorem again holds. Since $u(a, b) \in [0, 1]$, it follows from Theorem 2.8 that $u(a, 1) = u(1, b) = 1$. Now, by monotonicity we have

$$u(a, b) \leq u(a, 1) = u(1, b) = 1.$$

This concludes the proof. ■

Theorem 2.10. For all $a, b \in [0, 1]$, $i(a, b) \leq \min(a, b)$.

Proof: The proof of this theorem is similar to that of Theorem 2.8. First, we form the equation

$$i(a, i(1, 1)) = i(i(a, 1), 1)$$

based on the associativity of i. Then, using the boundary condition $i(1, 1) = 1$, we rewrite the equation as

$$i(a, 1) = i(i(a, 1), 1).$$

The only solution of this equation is $i(a, 1) = a$. Then, by monotonicity we have

$$i(a, b) \leq i(a, 1) = a,$$

and by commutativity

$$i(a, b) = i(b, a) \leq i(b, 1) = b,$$

which completes the proof. ■

Theorem 2.11. For all $a, b \in [0, 1]$, $i(a, b) \geq i_{\min}(a, b)$.

Proof: The proof is analogous to the proof of Theorem 2.9. When $b = 1$, then $i(a, b) = a$ (see the proof of Theorem 2.10) and the theorem is satisfied. Similarly, commutativity ensures that when $a = 1$, $i(a, b) = b$ and the theorem holds. Since $i(a, b) \in [0, 1]$, it follows from Theorem 2.10 that $i(a, 0) = i(0, b) = 0$. By monotonicity,

$$i(a, b) \geq i(0, b) = i(a, 0) = 0$$

and the proof is complete. ■

We can see that the standard max and min operations have a special significance: they represent, respectively, the lower bound of functions u (the strongest union) and the upper bound of functions i (the weakest intersection). Of all the possible pairs of fuzzy set unions and intersections, the max and min functions are closest to each other. That is, for all $a, b \in [0, 1]$, the inequality

$$\max(a, b) - \min(a, b) = |a - b| \leq u(a, b) - i(a, b)$$

is satisfied for any arbitrary pair of functions u and i that qualify as a fuzzy set union and intersection, respectively. The standard max and min operations therefore represent an extreme pair of all the possible pairs of fuzzy unions and intersections. Moreover, the functions of max and min are related to each other by DeMorgan's laws based on the standard complement $c(a) = 1 - a$, that is,

$$\max(a, b) = 1 - \min(1 - a, 1 - b),$$
$$\min(a, b) = 1 - \max(1 - a, 1 - b).$$

The operations $u_{\max}$ and $i_{\min}$ represent another pair of a fuzzy union and a fuzzy intersection which is extreme in the sense that for all $a, b \in [0, 1]$, the inequality

$$u_{\max}(a, b) - i_{\min}(a, b) \geq u(a, b) - i(a, b)$$

is satisfied for any arbitrary pair of fuzzy unions u and fuzzy intersections i. As is the case with the standard max and min operations, the operations $u_{\max}$ and $i_{\min}$ are related to each other by DeMorgan's laws under the standard complement, that is,

$$u_{\max}(a, b) = 1 - i_{\min}(1 - a, 1 - b),$$
$$i_{\min}(a, b) = 1 - u_{\max}(1 - a, 1 - b).$$

The Yager class of fuzzy unions and intersections, discussed in Secs. 2.3 and 2.4, covers the entire range of these operations as given by inequalities (2.8) and (2.9). The standard max and min operations are represented by $w \to \infty$, and the $u_{\max}$ and $i_{\min}$ operations at the other extreme are represented by $w \to 0$. Of the other classes of operations listed in Table 2.2, the full range of these operations is covered only by the Schweizer and Sklar class (with $p \to \infty$ for the standard operations and $p \to -\infty$ for the other extreme) and by the Dombi class (with $\lambda \to \infty$ representing the standard operations and $\lambda \to 0$ the other extreme).

The standard max and min operations are additionally significant in that they constitute the only fuzzy union and intersection operators that are continuous and idempotent. We express this fact by the following two theorems.

Theorem 2.12. $u(a, b) = \max(a, b)$ is the only continuous and idempotent fuzzy set union (i.e., the only function that satisfies Axioms u1 through u6).

Proof: By associativity, we can form the equation

$$u(a, u(a, b)) = u(u(a, a), b).$$

The application of idempotency (Axiom u6) allows us to replace $u(a, a)$ in this equation with a and thus to obtain

$$u(a, (u(a, b))) = i(a, b).$$

Similarly,

$$u(u(a, b), b) = u(a, u(b, b)) = u(a, b).$$

Hence,

$$u(a, u(a, b)) = u(u(a, b), b)$$

or, by commutativity,

$$u(a, u(a, b)) = u(b, u(a, b)). \qquad (2.10)$$

When $a = b$, idempotency is applicable and Eq. (2.10) is satisfied. Let $a < b$ and assume that $u(a, b) = \alpha$, where $\alpha \neq a$ and $\alpha \neq b$. Then, Eq. (2.10) becomes

$$u(a, \alpha) = u(b, \alpha).$$

Since u is continuous (Axiom u5) and monotonic nondecreasing (Axiom u3) with $u(0, \alpha) = \alpha$ and $u(1, \alpha) = 1$ (as determined in proofs of Theorems 2.8 and 2.9, respectively), there exists a pair $a, b \in [0, 1]$ such that

$$u(a, \alpha) < u(b, \alpha)$$

and, consequently, the assumption is not warranted.* Assume now that $u(a, b) = a = \min(a, b)$. This assumption is also unacceptable, since it violates the boundary conditions (Axiom u1) when $a = 0$ and $b = 1$. The final possibility is to consider $u(a, b) = b = \max(a, b)$. In this case, the boundary conditions are satisfied and Eq. (2.10) becomes

$$u(a, b) = u(b, b);$$

that is, it is satisfied for all $a < b$. Because of commutativity, the same argument can be repeated for $a > b$. Hence, max is the only function that satisfies Axioms u1 through u6. ■

Theorem 2.13. $i(a, b) = \min(a, b)$ is the only continuous and idempotent fuzzy set intersection (i.e., the only function that satisfies Axioms i1 through i6).

Proof: This theorem can be proven in exactly the same way as Theorem 2.12 by replacing function u with function i and by applying Axioms i1 through i6 instead of Axioms u1 through u6. The counterpart of Eq. (2.10) is

$$i(a, i(a, b)) = i(b, i(a, b)).$$

We use the same reasoning as in the proof of Theorem 2.8, albeit with different boundary conditions (Axiom i1 instead of Axiom u1) to conclude that $i(a, b) = \min(a, b)$ is the only solution of this equation. ■

* This is a consequence of the intermediate value theorem for continuous functions.

The operations of complement, union, and intersection defined on crisp subsets of X form a Boolean lattice on the power set $\mathcal{P}(X)$, as explained in Sec. 1.5; they possess the properties listed in Table 1.1 (or, in the abstracted form, in Table 1.4). The various fuzzy counterparts of these operations are defined on the power set $\tilde{\mathcal{P}}(X)$—the set of all *fuzzy* subsets of X. It is known that every possible selection of these three fuzzy operations violates some properties of the Boolean lattice on $\mathcal{P}(X)$. Different selected operations, however, may violate different properties of the Boolean lattice. Let us examine some possibilities.

It can be easily verified that the standard fuzzy operations satisfy all the properties of the Boolean lattice except the law of excluded middle $A \cup \bar{A} = X$ and the law of contradiction $A \cap \bar{A} = \varnothing$. These operations are said to form a *pseudo-complemented distributive lattice* on $\tilde{\mathcal{P}}(X)$. We know from Theorems 2.12 and 2.13 that the max and min operations are the only operations of fuzzy union and intersection that are idempotent. This means, in turn, that none of the other possible operations of fuzzy unions and intersections form a lattice on $\tilde{\mathcal{P}}(X)$. Some of them, however, satisfy the law of excluded middle and the law of contradiction, which for fuzzy sets have the form

$$u(a, c(a)) = 1 \quad \text{and} \quad i(a, c(a)) = 0$$

for all $a \in [0, 1]$. These latter operations are characterized by the following theorem.

Theorem 2.14. Fuzzy set operations of union, intersection, and continuous complement that satisfy the law of excluded middle and the law of contradiction are not idempotent or distributive.

Proof: Since the standard operations do not satisfy the two laws of excluded middle and of contradiction and, by Theorems 2.12 and 2.13, they are the only operations that are idempotent, operations that do satisfy these laws cannot be idempotent. Next, we must prove that these operations do not satisfy the distributive laws,

$$u(a, i(b, d)) = i(u(a, b), u(a, d)) \tag{2.11}$$

and

$$i(a, u(b, d)) = u(i(a, b), i(a, d)). \tag{2.12}$$

Let e denote the equilibrium of the complement c involved, that is, $c(e) = e$. Then, from the law of excluded middle, we obtain

$$u(e, c(e)) = u(e, e) = 1;$$

similarly, from the law of contradiction,

$$i(e, c(e)) = i(e, e) = 0.$$

Then, by applying e to the left hand side of Eq. (2.11), we obtain

$$u(e, i(e, e)) = u(e, 0).$$

We observe that e is neither 0 nor 1 because of the requirement that $c(0) = 1$ and $c(1) = 0$ (Axiom c1). By Theorem 2.8 and Theorem 2.9, we have $u(e, 0) = e$ and, consequently,

$$u(e, i(e, e)) = e \quad (\neq 1).$$

Now we apply e to the right hand side of Eq. (2.11) to obtain

$$i(u(e, e), u(e, e)) = i(1, 1) = 1.$$

This demonstrates that the distributive law (2.11) is violated.

Let us now apply e to the second distributive law (2.12). By Theorems 2.10 and 2.11, we obtain

$$i(e, u(e, e)) = i(e, 1) = e \quad (\neq 0),$$

and

$$u(i(e, e), i(e, e)) = u(0, 0) = 0,$$

which demonstrates that Eq. (2.12) is not satisfied. This completes the proof. ■

It follows from Theorem 2.14 that we may, if it is desired, preserve the law of excluded middle and the law of contradiction in our choice of fuzzy union and intersection operations by sacrificing idempotency and distributivity. The reverse is also true. The context of each particular application determines which of these options is preferable.

It is trivial to verify that u_{max}, i_{min}, and the standard complement satisfy the law of excluded middle and the law of contradiction. Another combination of operations of this type is the following:

$$u(a, b) = \min(1, a + b),$$

$$i(a, b) = \max(0, a + b - 1),$$

$$c(a) = 1 - a.$$

As previously mentioned, these operations do not form a lattice on $\tilde{\mathcal{P}}(X)$.

Given two of the three operations u, i, and c, it is sometimes desirable to determine the third operation in such a way that DeMorgan's laws are satisfied. This amounts to solving the functional equation

$$c(u(a, b)) = i(c(a), c(b)) \tag{2.13}$$

with respect to the unknown operation. When c is continuous and involutive, we have

$$u(a, b) = c[i(c(a), c(b))] \tag{2.14}$$

and

$$i(a, b) = c[u(c(a), c(b))]. \tag{2.15}$$

Example 2.1.

Given $u(a, b) = \max(a, b)$ and $c(a) = (1 - a^2)^{1/2}$, determine i such that DeMorgan's laws are satisfied. Employing Eq. (2.15), we obtain

$$i(a, b) = (1 - u^2[(1 - a^2)^{1/2}, (1 - b^2)^{1/2}])^{1/2}$$
$$= (1 - \max^2[(1 - a^2)^{1/2}, (1 - b^2)^{1/2}])^{1/2}.$$

Solving Eq. (2.13) for c is more difficult and may result in more than one solution. For example, if the standard max and min operations are employed for u and i, respectively, then every involutive complement satisfies the equation. Hence, max, min, and any of the Sugeno complements (or Yager complements) defined in Sec. 2.2 satisfy DeMorgan's laws.

For the sake of simplicity, we have omitted an examination of the properties of one operation that is important in fuzzy logic—fuzzy implication, $\Rightarrow$. This operation can be expressed in terms of fuzzy disjunction, $\vee$, fuzzy conjunction, $\wedge$, and negation, $^{-}$, by using the equivalences

$$a \Rightarrow b = \bar{a} \vee b \quad \text{or} \quad a \Rightarrow b = \overline{a \wedge \bar{b}}.$$

By employing the correspondences between logic operations and set operations defined in Table 1.5, the equivalences just given can be fully studied in terms of the functions

$$u(c(a), b) \quad \text{or} \quad c(i(a, c(b))).$$

Different fuzzy implications are obtained when different fuzzy complements c and either different fuzzy unions u or different fuzzy intersections i are used.

2.6 GENERAL AGGREGATION OPERATIONS

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined to produce a single set. In general, any *aggregation operation* is defined by a function

$$h : [0, 1]^n \rightarrow [0, 1]$$

for some $n \geq 2$. When applied to n fuzzy sets $A_1, A_2, \ldots, A_n$ defined on X, h produces an aggregate fuzzy set A by operating on the membership grades of each $x \in X$ in the aggregated sets. Thus,

$$\mu_A(x) = h(\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x))$$

for each $x \in X$.

In order to qualify as an aggregation function, h must satisfy at least the following two axiomatic requirements, which express the essence of the notion of aggregation:

Axiom h1. $h(0, 0, \ldots, 0) = 0$ and $h(1, 1, \ldots, 1) = 1$ (*boundary conditions*).

Axiom h2. For any pair $(a_i \mid i \in \mathbb{N}_n)$ and $(b_i \mid i \in \mathbb{N}_n)$, where $a_i \in [0, 1]$ and $b_i \in [0, 1]$, if $a_i \geq b_i$ for all $i \in \mathbb{N}_n$, then $h(a_i \mid i \in \mathbb{N}_n) \geq h(b_i \mid i \in \mathbb{N}_n)$, that is, h is *monotonic nondecreasing* in all its arguments.

Two additional axioms are usually employed to characterize aggregation operations despite the fact that they are not essential:

Axiom h3. h is a *continuous* function.

Axiom h4. h is a *symmetric* function in all its arguments, that is,

$$h(a_i \mid i \in \mathbb{N}_n) = h(a_{p(i)} \mid i \in \mathbb{N}_n)$$

for any permutation p on $\mathbb{N}_n$.

Axiom h3 guarantees that an infinitesimal variation in any argument of h does not produce a noticeable change in the aggregate. Axiom h4 reflects the usual assumption that the aggregated sets are equally important. If this assumption is not warranted in some application context, the symmetry axiom must be dropped.

We can easily see that fuzzy unions and intersections qualify as aggregation operations on fuzzy sets. Although they are defined for only two arguments, their property of associativity guaranteed by Axioms u4 and i4 provides a mechanism for extending their definition to any number of arguments. Hence, fuzzy unions and intersections can be viewed as special aggregation operations that are symmetric, usually continuous, and required to satisfy some additional boundary conditions. As a result of these additional requirements, fuzzy unions and intersections can produce only aggregates that are subject to restrictions (2.8) and (2.9). In particular, they do not produce any aggregates of $a_1, a_2, \ldots, a_n$ that produce values between $\min(a_1, a_2, \ldots, a_n)$ and $\max(a_1, a_2, \ldots, a_n)$. Aggregates that are not restricted in this way are, however, allowed by Axioms h1 through h4; operations that produce them are usually called *averaging operations*.

Averaging operations are therefore aggregation operations for which

$$\min(a_1, a_2, \ldots, a_n) \leq h(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n). \quad (2.16)$$

In other words, the standard max and min operations represent boundaries between the averaging operations and the fuzzy unions and intersections, respectively.

One class of averaging operations that covers the entire interval between the min and max operations consists of *generalized means*. These are defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha}, \quad (2.17)$$

where $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$ is a parameter by which different means are distinguished.

Function h_α clearly satisfies Axioms h1 through h4 and, consequently, it represents a parameterized class of continuous and symmetric aggregation operations. It also satisfies the inequalities (2.16) for all $\alpha \in \mathbb{R}$, with its lower bound

$$h_{-\infty}(a_1, a_2, \ldots, a_n) = \min(a_1, a_2, \ldots, a_n)$$

and its upper bound

$$h_{\infty}(a_1, a_2, \ldots, a_n) = \max(a_1, a_2, \ldots, a_n).$$

For fixed arguments, function h_α is monotonic increasing with α. For $\alpha \to 0$, the function h_α becomes the *geometric mean*

$$h_0(a_1, a_2, \ldots, a_n) = (a_1 \cdot a_2 \cdots a_n)^{1/n};$$

furthermore,

$$h_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}$$

is the *harmonic mean* and

$$h_1(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

is the *arithmetic mean*.

Since it is not obvious that h_α represents the geometric mean for $\alpha \to 0$, we use the following theorem.

Theorem 2.15. Let h_α be given by Eq. (2.17). Then,

$$\lim_{\alpha \to 0} h_\alpha = (a_1 \cdot a_2 \cdots a_n)^{1/n}.$$

Proof: First, we determine

$$\lim_{\alpha \to 0} \ln h_\alpha = \lim_{\alpha \to 0} \frac{\ln(a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha) - \ln n}{\alpha}.$$

Using l'Hospital's rule, we now have

$$\lim_{\alpha \to 0} \ln h_\alpha = \lim_{\alpha \to 0} \frac{a_1^\alpha \ln a_1 + a_2^\alpha \ln a_2 + \cdots + a_n^\alpha \ln a_n}{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}$$

$$= \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} = \ln(a_1 \cdot a_2 \cdots a_n)^{1/n}.$$

Hence,

$$\lim_{\alpha \to 0} h_\alpha = (a_1 \cdot a_2 \cdots a_n)^{1/n}. \quad \blacksquare$$

When it is desirable to accommodate variations in the importance of individual aggregated sets, the function h_α can be generalized into *weighted generalized*

means, as defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n; w_1, w_2, \ldots, w_n) = \left(\sum_{i=1}^{n} w_i a_i^\alpha\right)^{1/\alpha}, \qquad (2.18)$$

where $w_i \geq 0$ ($i \in \mathbb{N}_n$) are weights that express the relative importance of the aggregated sets; it is required that

$$\sum_{i=1}^{n} w_i = 1.$$

The weighted means are obviously not symmetric. For fixed arguments and weights, the function h_α given by Eq. (2.18) is monotonic increasing with α.

The full scope of fuzzy aggregation operations is summarized in Fig. 2.5. Included in this diagram are only the generalized means, which cover the entire range of averaging operators, and those parameterized classes of fuzzy unions and intersections given in Table 2.2 that cover the full ranges specified by the inequalities (2.8) and (2.9). For each class of operators, the range of the respective parameter is indicated. Given one of these families of operations, the identification of a suitable operation for a specific application is equivalent to the estimation of the parameter involved.

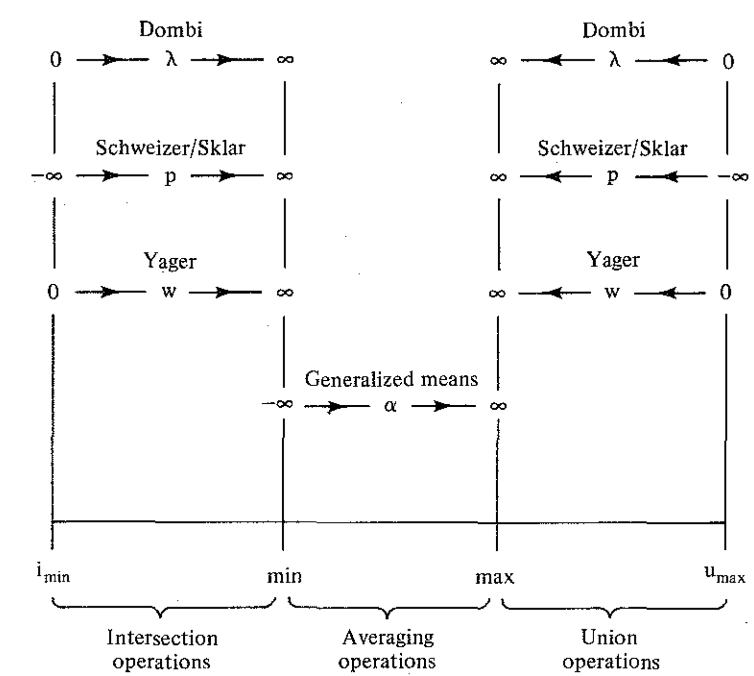


Figure 2.5. The full scope of fuzzy aggregation operations.

NOTES

2.1. In the seminal paper by Zadeh [1965a], fuzzy set theory is formulated in terms of the standard operations of complement, union, and intersection, but other possibilities of combining fuzzy sets are also hinted at.

2.2. The first axiomatic treatment of fuzzy set operations was presented by Bellman and Giertz [1973]. They demonstrated the uniqueness of the max and min operators in terms of axioms that consist of our axiomatic skeletons for u and i, the axioms of continuity, distributivity, strict increase of $u(a, a)$ and $i(a, a)$ in a, and lower and upper bounds $u(a, b) \geq \max(a, b)$ and $i(a, b) \leq \min(a, b)$. They concluded, however, that the operation of fuzzy complement is not unique even when all reasonable requirements (boundary conditions, monotonicity, continuity, and involution) are employed as axioms. A thorough investigation of properties of the max and min operators was done by Voxman and Goetschel [1983].

2.3. The Sugeno class of fuzzy complements results from special measures (called λ-measures) introduced by Sugeno [1977]. The Yager class of fuzzy complements is derived from his class of fuzzy unions defined by Eq. (2.6) by requiring that $A \cup C(A) = X$, where A is a fuzzy set defined on X. This requirement can be expressed more specifically by requiring that $u_w(a, c_w(a)) = 1$ for all $a \in [0, 1]$ and all $w > 0$.

2.4. Different approaches to the study of fuzzy complements were used by Lowen [1978], Esteva, Trillas, and Domingo [1981], and Ovchinnikov [1981, 1983]. Yager [1979b, 1980a] investigated fuzzy complements for the purpose of developing useful measures of fuzziness (Sec. 5.2). Our presentation of fuzzy complements in Sec. 2.2 is based upon a paper by Higashi and Klir [1982], which is also motivated by the aim of developing measures of fuzziness.

2.5. The Yager class of fuzzy unions and intersections was introduced in a paper by Yager [1980b], which contains some additional characteristics of these classes. Yager [1982b] also addressed the question of the meaning of the parameter w in his class and the problem of selecting appropriate operations for various purposes.

2.6. The axiomatic skeletons that we use for characterizing fuzzy intersections and unions are known in the literature as *triangular norms* (or *t-norms*) and *triangular conorms* (or *t-conorms*), respectively [Schweizer and Sklar, 1960, 1961, 1983]. These concepts were originally introduced by Menger [1942] in his study of statistical metric spaces. In current literature on fuzzy set theory, the terms t-norms and t-conorms are used routinely.

2.7. Classes of functions given in Table 2.2 that can be employed for fuzzy unions and intersections were proposed by Schweizer and Sklar [1961, 1963, 1983], Hamacher [1978], Frank [1979], Yager [1980b], Dubois and Prade [1980b], and Dombi [1982]. Additional theoretical studies of fuzzy set operations were done by Trillas, Alsina, and Valverde [1982], Czogała and Drewniak [1984], Klement [1984], and Silvert [1979]. A good overview of the various classes of fuzzy set operations was prepared by Dubois and Prade [1982a]; they also overviewed properties of various combinations of fuzzy set operations [Dubois and Prade, 1980b].

2.8. The issue of which operations on fuzzy sets are suitable in various situations was studied by Zimmermann [1978a], Thole, Zimmermann, and Zysno [1979], Zimmermann and Zysno [1980], Yager [1979a, 1982b], and Dubois and Prade [1980b].

2.9. One class of operators not covered in this book are fuzzy implication operators. They were extensively studied by Bandler and Kohout [1980a, b] and Yager [1983b].

2.10. An excellent overview of the whole spectrum of aggregation operations on fuzzy sets was prepared by Dubois and Prade [1985a]; it covers fuzzy unions and intersections as well as averaging operations. In another paper, Dubois and Prade [1984b] presented a similar overview in the context of decision-making applications. The class of generalized means defined by Eq. (2.17) is covered in a paper by Dyckhoff and Pedrycz [1984].

EXERCISES

2.1. Using Sugeno complements for $\lambda = 1, 2, 10$ and Yager complements for $w = 1, 2, 3$, determine complements of the following fuzzy sets:
(a) the fuzzy number defined in Fig. 1.2;
(b) the fuzzy sets defined in Exercise 1.3;
(c) some of the fuzzy sets defined in Fig. 1.9.

2.2. Does the function $c(a) = (1 - a)^w$ qualify for each $w > 0$ as a fuzzy complement? Plot the function for some values $w > 1$ and some values $w < 1$.

2.3. Prove that the Sugeno complements are monotonic nonincreasing (Axiom c2) for all $\lambda \in (-1, \infty)$.

2.4. Show that the Sugeno complements are involutive for all $\lambda \in (-1, \infty)$. Show that the Yager complements are involutive for $w \in (1, \infty)$.

2.5. Show that the equilibria e_{c_w} for the Yager fuzzy complements are given by the formula

$$e_{c_w} = (1/2)^{1/w}.$$

Plot this function for $w \in (0, 10]$.

2.6. Prove that Axioms u1 through u5 (or i1 through i5) are satisfied by all fuzzy unions (or intersections) in the Yager class.

2.7. Prove that the following properties are satisfied by all fuzzy unions in the Yager class:
(a) $u_w(a, 0) = a$; (b) $u_w(a, 1) = 1$;
(c) $u_w(a, a) \geq a$; (d) if $w \leq w'$, then $u_w(a, b) \geq u_{w'}(a, b)$;
(e) $\lim_{w \to 0} u_w(a, b) = u_{\max}(a, b)$.

2.8. Prove that the following properties are satisfied by all fuzzy intersections in the Yager class:
(a) $i_w(a, 0) = 0$; (b) $i_w(a, 1) = a$;
(c) $i_w(a, a) \leq a$; (d) if $w \leq w'$, then $i_w(a, b) \leq i_{w'}(a, b)$;
(e) $\lim_{w \to 0} i_w(a, b) = i_{\min}(a, b)$.

2.9. Show that $u_w(a, c_w(a)) = 1$ for all $a \in [0, 1]$ and all $w > 0$, where u_w and c_w denote the Yager union and complement, respectively (Note 2.3).

2.10. For each class of fuzzy set unions and intersections defined in Table 2.2 and several values of the parameter involved (values 1 and 2, for instance), determine membership functions of the respective unions and intersections in a form similar to Table 2.1.

2.11. For each of the classes of fuzzy unions defined by the parameterized functions in Table 2.2, show that the function decreases with an increase in the parameter.

2.12. For each of the classes of fuzzy intersections defined by the parameterized functions in Table 2.2, show that the function increases with any increase in the parameter.

2.13. The proof of Theorem 2.13 is outlined in Sec. 2.5. Describe the proof in full detail.

2.14. Show that the following operations satisfy the law of excluded middle and the law of contradiction:
(a) $u_{\max}$, $i_{\min}$, $c(a) = 1 - a$;
(b) $u(a, b) = \min(1, a + b)$, $i(a, b) = \max(0, a + b - 1)$, $c(a) = 1 - a$.

2.15. Show that the following operations on fuzzy sets satisfy DeMorgan's laws:
(a) $u_{\max}$, $i_{\min}$, $c(a) = 1 - a$;
(b) max, min, c_λ is a Sugeno complement for some $\lambda \in (-1, \infty)$;
(c) max, min, c_w is a Yager complement for some $w \in (0, \infty)$;

2.16. Determine the membership function based on the generalized means (in a form similar to Table 2.1) for $\alpha = -2, -1, 0, 1, 2$; assume only two arguments and, then, repeat one of the cases for three arguments.

2.17. Show that the generalized means defined by Eq. (2.17) become min and max operations for $\alpha \to -\infty$ and $\alpha \to \infty$, respectively.

2.18. Demonstrate that the generalized means h_α defined by Eq. (2.17) are monotonic increasing with α for fixed arguments.

3

FUZZY RELATIONS

3.1 CRISP AND FUZZY RELATIONS

A *crisp relation* represents the presence or absence of association, interaction, or interconnectedness between the elements of two or more sets. This concept can be generalized to allow for various degrees or strengths of relation or interaction between elements. Degrees of association can be represented by membership grades in a *fuzzy relation* in the same way as degrees of set membership are represented in the fuzzy set. In fact, just as the crisp set can be viewed as a restricted case of the more general fuzzy set concept, the crisp relation can be considered to be a restricted case of the fuzzy relation.

Throughout this chapter the concepts and properties of crisp relations are briefly discussed as a refresher and in order to demonstrate their generalized application to fuzzy relations.

The *Cartesian product* of two crisp sets X and Y, denoted by $X \times Y$, is the crisp set of all ordered pairs such that the first element in each pair is a member of X and the second element is a member of Y. Formally,

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Note that if $X \neq Y$, then $X \times Y \neq Y \times X$.

The Cartesian product can be generalized for a family of crisp sets $\{X_i \mid i \in \mathbb{N}_n\}$ and denoted either by $X_1 \times X_2 \times \cdots \times X_n$ or by $\times_{i \in \mathbb{N}_n} X_i$. Elements of the Cartesian product of n crisp sets are n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$ for all $i \in \mathbb{N}_n$. Thus,

$$\underset{i \in \mathbb{N}_n}{\times} X_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \quad \text{for all} \quad i \in \mathbb{N}_n\}.$$