Time Scale Discrete Fourier Transforms: Search Homework Problem

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***Abstract*—The discrete and continuous Fourier transforms are applicable to discrete and continuous time signals respectively. Time scales allows generalization to to any closed set of points on the real line. Discrete and continuous time are special cases. Using the Hilger exponential from time scale calculus, the discrete Fourier transform (DFT) is extended to signals on a set of points with arbitrary spacing. We wonder what good terms are to find good discrete values in the frequency domain.**

I. INTRODUCTION

A time scale is any collection of closed points on the real line. Continuous time, $\mathbb{R}$, and discrete time $\mathbb{Z}$, are special cases. The calculus of time scales was introduced by Hilger [11]. Time scales have found utility in describing the behavior of dynamic systems [1], [13] and have been applied to control theory [3], [4], [5], [7], [10].

On $\mathbb{R}$ and $\mathbb{Z}$, respectively, the continuous time and discrete time Fourier transforms are well studied [16]. Properties of the Laplace and Fourier transforms on time scales have been extended to time scales with unbounded domains [1], [6], [8], [9], [12], [14], [16].

The conventional *discrete Fourier transform* (DFT) is defined over a finite number of uniformly spaced points. This paper extends the DFT to a finite number of discrete time points that are not uniformly spaced.[1] The time scale of a finite number of N discrete points, $\mathbb{D}_N$, is shown to uniquely map into a frequency scale (a time scale in the frequency domain), $\mathbb{U}_N$, in the Fourier domain. Familiar Fourier transform theorems, including the shift, convolution and derivative theorems, are shown to generalize to the *time scale* DFT (TS-DFT).

II. TIME SCALES

Our introduction to time scales is limited to that needed to establish notation. A more detailed explanation are in our previous papers [4], [5], [6], [8], [9], [10], [13], [14], [16] and a complete rigorous treatment is in the text by Bohner and Peterson [1].

1) A *time scale*, $\mathbb{T}$, is any collection of closed intervals on the real line. Generally, the time scale can contain both discrete time points and continuous time intervals. Since our development of TS-DFT is only on time scales containing discrete points, we henceforth restrict attention to time scales containing discrete points,[2] denoted $\mathbb{D}$. Discrete time, $\mathbb{Z}$, is a special case.
2) The *graininess*, $\mu(t)$, is the distance between adjacent points in a time scale at time $t \in \mathbb{T}$ and is defined generally by
$$\mu(t) = \left(\inf_{\tau > t, \tau \in \mathbb{T}} \tau \right) - t.$$
For $\mathbb{D}$,
$$\mu(t_n) = t_{n+1} - t_n.$$
3) The *Hilger derivative* of a function $x(t)$ at $t \in \mathbb{T}$ is
$$x^{\Delta}(t) := \frac{x(t+\mu(t)) - x(t)}{\mu(t)}.$$
When $\mu(t) = dt \; (= 0)$, the Hilger derivative is interpreted in the limiting sense and
$$x^{\Delta}(t) = \frac{d}{dt} x(t).$$
For $\mathbb{D}$, we have
$$x^{\Delta}(t_n) = \frac{x(t_{n+1}) - x(t_n)}{\mu(t_n)}.$$
4) If $y(t) = x^{\Delta}(t)$, then the *definite time scale integral* is
$$\int_a^b y(t) \Delta t = x(b) - x(a).$$
For $\mathbb{D}$, we have [1]
$$\int_{t_p}^{t_q} y(t) \Delta t = \sum_{n=p}^{q-1} y(t_n) \mu(t_n).$$
5) When $x(0) = 1$, the solution to the *Hilger differential equation*,
$$x^{\Delta}(t) = z x(t),$$
is $x(t) = e_z(t)$ where the *generalized exponential* is
$$e_z(t) := \exp \left(\int_{\tau=0}^{t} \frac{\ln(1 + z\mu(\tau))}{\mu(\tau)} \Delta\tau \right).$$

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[1] Our development is distinct from the time scale Fourier transform proposed by Hilger [12], [14], [16]. Our treatment more closely resembles Laplace transform generalizations where two signals on a time scale $\mathbb{T}$, when convolved, result in a signal on the same time scale, $\mathbb{T}$ [1], [6], [8], [9].

[2] We use $\mathbb{D}$ to denote a time scale with an arbitrary, possibly infinite, set of discrete isolated points. The notation $\mathbb{D}_N$ indicates the time scale has N points.

For $\mathbb{D}$ and $n > 0$,

$$e_z(t_n) = \prod_{m=0}^{n-1} \left(1 + z\mu(t_m)\right). \tag{1}$$

Since $\mu(t_m)$ is real,

$$e_z^*(t_n) = e_{z^*}(t_n) \tag{2}$$

The properties of the generalized exponential parallel those of z^n for the z-transform and $e^{j\omega t}$ for the Fourier transform are responsible for the utility of the TS-DFT.

Here are some examples of discrete time scales. All n indices run from 0 to N.

- Uniform time scale, $\mathbb{Z}$.

  $$t_n = n$$

- Log time scale.

  $$t_n = \log(n+1)$$

- Harmonic time scale.

  $$t_0 = 0.$$

  Otherwise

  $$t_n = \sum_{m=1}^{n} \frac{1}{m}.$$

- Random time scale.

  $$t_0 = 0.$$

  Otherwise, random points chosen on a fixed interval.

A. Discrete Time Scale Fourier Transform

Let

$$X(z_k) = \sum_{n=0}^{N-1} x(t_n) e_{\ominus z_k}(t_n^\sigma)\mu(t_n) \tag{3}$$

where $t_n^\sigma := t_{n+1}$ and $1 \le k \le N$. The values of z_k have yet to be determined. Equation (3) can be considered a matrix vector multiplication.

$$\vec{X} = \mathbf{D}\vec{x}$$

where the matrix $\mathbf{D}$ has elements

$$[\mathbf{D}]_{kn} = e_{\ominus z_k}(t_n^\sigma)\mu(t_n)$$

Thus the inverse can be found from

$$\vec{x} = \mathbf{D}^{-1}\vec{X}a$$

The condition of $\mathbf{D}$ is

$$\text{cond } [\mathbf{D}] = \frac{|\lambda_{max}|}{|\lambda_{min}|}.$$

To make sure $\mathbf{D}$ is easily inverted, we need it's condition to be small. It's condition is a function of the choice of the z_k's.

The z_k's can be complex, but if there is a complex number, then the complex conjugate of z_k must also be included.

The search problem is to find the z_k's such that the condition of $\mathbf{D}$ is minimum. Do this for $\mathbb{Z}$ and the time scales

Find the optimal values of z_k for the uniform time scale, the log time scale, the harmonic time scale, and a random time scale. Start with $N = 4$ and then $N = 8$. I'll be impressed if you do $N = 16$.

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