Jawdropping Probability: The Two Envelope Problem & Bermoulli's Wager

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Here are two astonishing problems in elementary probability.

1 The Two Envelope Problem

You are given two envelopes. Inside each is a slip of paper with two numbers written on them. The two numbers are not the same. The way the numbers are chosen is unknown. You chose one of the envelopes at random. You open it and look at the number. You must make a choice. Either the number you hold is the largest number or the number in unopened envelope is and you have to make a decision.

If your choice is the largest of the two numbers, you get a million dollars. If it isn't, you get nothing.

At first flush, it looks like anything you do won't help you win beyond a 50-50 chance. It's not true. You can always make a decision that gives you a strictly better than a 50-50 chance. It may not be much better, but the chances of winning are always in your favor

1.1 The Algorithm

First, choose a squashing function, $g(x)$, as shown in Figure 1, that takes any point between $-\infty$ and ∞ and maps it to the interval [0,1]. The function must be strictly increasing but is otherwise arbitrary. Examples are the logistic function

$$g(x) = \frac{1}{1 + e^{-(\mu - x)/\sigma}}$$

and

$$g(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-\mu}{\sigma}\right)$$

where μ is an arbitrarily chosen shift parameter and $\sigma > 0$ is and arbitrarily chosen scaling parameter. The cumulative distribution function, $F_Z(x)$, of any random variable can serve as $g(x)$ if the domain of Z is $(-\infty, \infty)$. Since $g(x)$ is strictly increasing, we can be assured that, if $B > A$, then $g(A) > g(B)$.

Here is how you win. Choose one of the two envelopes at random and and open it. It contains the number C. Let $c = g(C)$. Generate a random variable X that is uniform on the interval [0,1].

- If $X > c$, then declare the c as your biggest number.
- If $X < c$, then declare the contents of unopened envelope as your biggest number.

Your chance of choosing the correct envelope will be greater than one half.

1.1.1 Proof

Let the two envelopes be $\mathbb{A}$ and $\mathbb{B}$. Assume with no loss of generality, that envelope $\mathbb{B}$ contains the biggest number so that $B > A$ and $b > a$ where $a = g(A)$ and $b = g(B)$. Let p denote your probability of success. Then, imposing independence,

$$\begin{aligned} p &= \Pr\left[X < b,\ \text{you chose envelope } \mathbb{B}\right] \\ &\quad + \Pr\left[X > a,\ \text{you chose envelope } \mathbb{A}\right] \\ &= \frac{1}{2}b + \frac{1}{2}(1-a) \\ &= \frac{1}{2} + \frac{1}{2}(b-a) > \frac{1}{2}. \end{aligned}$$

Since we are assured that $b - a > 0$, we are therefore assured that

$$p > \frac{1}{2}.$$

This is astonishing and counterintuitive.

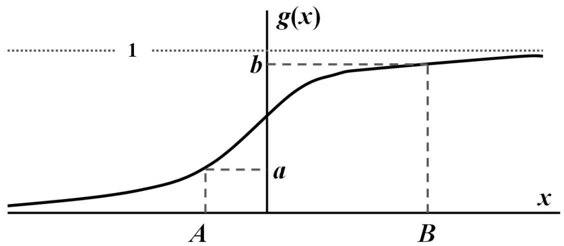

Figure 1: b

1.2 Thoughts

- If you have a two randomly chosen bits, can you predict which one is bigger given only one of them? In a way. If you have a 1, the chance of zero of increasing and, for a zero, no chance of decreasing.

- Give a random digit from π, can we predict whether the subsequent or previous values are larger or smaller? Of course. If the digit is a 9, for example, the next digit will be smaller. If its zero, higher.

- Any stationary stochastic process is similar. It has a constant mean. If a value is above the mean, chances are it will lower and, below the mean, higher. Financial technical analysts call this *mean reversion*.

- We can't use the two envelope problem to get rich on the stock market. Predicting whether tomorrow's stock price will be higher are lower given today's stock price violates the premise of the algorithm. The algorithm requires us to choose today's price or tomorrow's price at random. We can't choose tomorrow's price as one of the two possibilities.

  Indeed, we can construct a random walk time series. It time n, the value of the process is $X[n] = N$. At time $n+1$ the value is either $X[n+1] = N+1$ or $X[n+1] = N-1$ with equal probability. By construction, this process will not work using the two envelope solution in the sense that nothing known about $X[n]$ can tell you with other than a 50-50 whether the subsequent value is larger or smaller.

2 Bernoulli's Wager

You walk into a casino. There's a game table called *The Coin Flip Game* that pays you a lot of money for flipping a large number of heads using a fair coin. You flip the coin until you get a tails. Here's the payoff.

If you get your first tails on the	first	flip, you get \$2
If you get your first tails on the	second	flip, you get $\$4 = \$2^2$
If you get your first tails on the	third	flip, you get $\$8 = \$2^3$
$\vdots$	$\vdots$	$\vdots$
If you get your first tails on the	n^{th}	flip, you get $\$2^n$

How much would you pay to play this game?

At least \$2 because you always win at least \$2 no matter what the outcome is. How about \$10? How about \$100,000 ?

Gambling houses decide winnings using the expected value of winning. Using the law of large numbers, they know that winning in games of chance will always approach the mean of the underlying random variable. So let's compute the odds of winning *The Coin Flip Game*. Let your winnings be X. Then the expected value of X will allow us to set the odds if we are the gambling house, or determine how much we would pay to play the game.

The expected value of X is

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} \$2^n \times \Pr[\text{ getting the first tails on the } n^{th} \text{ flip }] \\
&= \$2 \times \frac{1}{2} + \$4 \times \frac{1}{4} + \$8 \times \frac{1}{8} + \cdots + \$2^n \times \frac{1}{2^n} + \cdots \\
&= 1 + 1 + 1 + \cdots + 1 + \cdots \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

The expected winnings off of this game is therefore infinite! There should therefore be no price you won't pay to play.

But would you pay a billion dollars to play *The Coin Flip Game*? I wouldn't.

So what's wrong here? Here's a hint. It would take more pocket money than is in the Obama deficit to make your fortune playing *The Coin Flip Game*.